Thermal development of high Prandtl channel flows of single-mode linearised Phan-Thien—Tanner fluids

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Introduction

Polymer melts are very viscous fluids that can often be accurately described in shear flow by a single mode simplified linearised Phan-Thien—Tanner (SPTT) constitutive equation [1,2]. Of practical relevance, as in mold filling, is the thermal development of this fluid in a channel which takes place for a condition of fully-developed hydrodynamic inlet flow because the Prandtl number is well in excess of 1000. This is known as the Graetz problem and has been solved in the past for Newtonian fluids [3], with viscous dissipation [4], for power law fluids [5], but not for viscoelastic fluids.

In this work, the Graetz problem for the channel flow of an SPTT with constant wall temperature and viscous dissipation is presented.

Governing Equations

The channel is aligned with the *x*-axis, *y* is the transverse coordinate and *H* is its half-width. The flow is fully-developed at inlet with local velocity *u(y)* and bulk velocity $\overline{U}$. The rheological equation determines the stress field and is given [1] by:

$$\left(1+\frac{\lambda\varepsilon}{\eta}\tau_{kk}\right)\tau_{ij}+\lambda\tau_{ij}^{(1)}=\eta\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \qquad (1)$$

where $\tau_{ij}^{(1)}$ is Oldroyd's upper convected derivative, λ is the relaxation time, η is the viscosity coefficient and ε is an elongational parameter.

Oliveira and Pinho [6] solved, for this fluid, the momentum and continuity equations to give the fully-developed hydrodynamic channel flow

$$u^*=\frac{u}{\overline{U}}=\frac{3}{2}\frac{\overline{U}_N}{\overline{U}}\left(1-y^{*2}\right)\left[1+a\left(1+y^{*2}\right)\right] \qquad (2)$$

$$\tau_{xy}^*=\frac{\tau_{xy}}{\eta\overline{U}/H}=-3\frac{\overline{U}_N}{\overline{U}}y^* \qquad (3)$$

where $y^*=y/H$. So, starred quantities are nondimensional and $\overline{U}_N$ represents the bulk velocity for the Newtonian fluid subjected to the same pressure gradient (i.e. $\overline{U}_N=-H^2(dp/dx)/(3\eta)$) thus defining the dimensionless pressure gradient $\chi\equiv\overline{U}_N/\overline{U}$ given as the solution of the following cubic equation

$$\frac{54}{5}\varepsilon We^2\left(\frac{\overline{U}_N}{\overline{U}}\right)^3+\frac{\overline{U}_N}{\overline{U}}=1 \qquad (4)$$

Flow viscoelasticity is usually characterised by the Weissenberg number $We\equiv\lambda\overline{U}/H$. Here, elastic and elongational characteristics are accounted for together in nondimensional parameter:

$$a=9\varepsilon We^2\left(\frac{\overline{U}_N}{\overline{U}}\right)^2 \qquad (5)$$

Following previous work [5,7] the Fourier law for heat conduction is assumed valid for the PTT viscoelastic fluid and all fluid properties are constant and independent of temperature. The energy conservation equation is written in non-dimensional form as

$$u^*\frac{\partial\theta}{\partial x^*}=\frac{\partial}{\partial y^*}\left(\frac{\partial\theta}{\partial y^*}\right)+Br\,\tau_{xy}^*\frac{du^*}{dy^*} \qquad (6)$$

where $x^*\equiv x\alpha/R^2\overline{U}$ and α is the thermal diffusivity. The temperature and the Brinkman

number (this term accounts for viscous dissipation) are defined as

$$\theta \equiv \frac{T - T_i}{T_w - T_i} \quad (7)$$

$$Br \equiv \frac{\eta \overline{U}^2}{k(T_w - T_i)} \quad (8)$$

where T_w and T_i are the wall and inlet bulk temperatures. The boundary conditions are: Inlet:

$$\theta(y^*, 0) = 0 \quad (9)$$

Symmetry plane: $\frac{\partial \theta(0, x^*)}{\partial y^*} = 0$ (10)

Wall: $\theta(1, x^*) = 1$ (11)

Method of Solution

The set of equations (6,9-11) is similar to that found in the classical Graetz problem, albeit more complicate due to the inclusion of viscous dissipation, and the fact that the velocity profile for the viscoelastic fluid is not as simple as that for the Newtonian fluid. The separation solution method explained by Mikhailov and Ösizik [8] is adopted here and leads to an eigenvalue problem. The method is particularly appropriate for an nonhomogeneous problem like the one at hand, and gives the solution of the temperature distribution $\theta(y^*, x^*)$ as a sum of three terms: a particular solution of the nonhomogeneous energy equation (due to viscous dissipation), a term related to the asymptotic (long *x*) transverse distribution of temperature and the general solution of the homogeneous energy balance (setting *Br*= 0) which becomes an infinite series of eigenvalues and eigenfunctions of the corresponding Sturm-Liouville problem.

We shall omit the full details of the derivation which are very cumbersome and limit ourselves to giving the final expressions for the solution in terms of the temperature distribution and the Nusselt number variation. These expressions are in terms of eigenvalues μ_i and functions y_1 to y_4, and the numerical methods used for these calculations are briefly explained next.

Eigen value problem

The solution of Eq. (6) is the sum of the general solution of the corresponding homogeneous equation with a particular solution of the equation. To solve the homogeneous equation, separation of variables is used leading to

$$\theta(y^*, x^*) = \psi(y^*)\phi(x^*) \quad (12)$$

where

$$\frac{d\phi(x^*)}{dx^*} + \mu^2 \phi(x^*) = 0 \quad (13)$$

$$\frac{d}{dy^*}\left(\frac{d\psi(\mu, y^*)}{dy^*}\right) + \mu^2 u^* \psi(\mu, y^*) = 0 \quad (14)$$

Eq. (13) is readily integrated and gives rise to the inverse exponential function on x^*, but depends on the eigenvalues μ^2 that are determined as part of the solution of Eq. (14). Eq. (14) is a Sturm-Liouville type equation, the solution of which is subject to boundary conditions

$$y^* = 0, \frac{d\psi(\mu, y^*)}{dy^*} = 0 \quad (15)$$

$$y^* = 1, \psi(\mu, y^*) = 0 \quad (16)$$

The eigenvalues for this problem have been evaluated numerically by means of a freeware Fortran code SLEDGE (Pruess and Fulton, Netlib, cited by Pryce [9]. This code solves the general problem:

$$\frac{d}{dX}\left(X^n \frac{dY}{dX}\right) + \mu^2 f(X) Y = 0 \quad (17)$$

and provides very accurate results for the eigenvalues. The calculated eigenvalues were accurate to at least 12 significant digits and for accurate thermal results, in each case the infinite series was only truncated at 161th term. In this problem, Eq. (6) depends on εWe^2 via parameter *a*, and on the mean velocity ratio χ. Thus, everytime those parameters are changed the equations must be numerically solved to calculate a new set of eigenvalues. Clearly the amount of work is greatly enhanced on passing from the Newtonian to the viscoelastic case, in view of the increased number of free parameters.

Once the eigenvalues are known, the normalised temperature distribution and the Nusselt number can be determined which also depends on four functions $y_k(\mu, y^*)$ (*k*=1 to 4) and its derivatives [8]. To determine these functions four ODE's had

to be numerically solved by means of a standard ODE solver based on a fourth-order Runge-Kutta method.

Results

The temperature distribution is given by:

$$\theta\left(y^*,x^*\right)=1-\frac{12a\left(y^{*6}-1\right)+15\left(y^{*4}-1\right)}{20}Br\chi^2$$
$$+2\times\sum_{i=1}^{\infty}\frac{e^{-\mu_i^2x^*}y_2\left(\mu_i,y^*\right)}{\mu_i y_4\left(\mu_i,1\right)y_1\left(\mu_i,1\right)}\left\{y_1\left(\mu_i,1\right)-9Br\times\right.$$
$$\left.\int_0^1\chi^2y^{*2}\left(2ay^{*2}+1\right)y_2\left(\mu_i,y^*\right)dy^*\right\} \quad (18)$$

where the first two terms correspond to the asymptotic solution ($x^*\rightarrow\infty$) with viscous dissipation and the last term is the infinite series resulting from the Sturm-Liouville problem with eigenvalues μ_i^2. In engineering calculations one is generally more interested in knowing how the heat flux at the duct wall varies with the axial distance. This can be expressed in a nondimensional way by means of a local Nusselt number defined as:

$$Nu\equiv\frac{hD_H}{k}=\frac{4}{1-\theta_b}\left(\frac{\partial\theta}{\partial y^*}\right)_{y^*=1} \quad (19)$$

where $D_H=4H$. On calculating θ_b and the temperature derivative we arrive at

$$Nu\left(x^*\right)=\frac{-12Br\chi+8S_1}{\frac{-24\left(54a^2+110a+55\right)}{1925}Br\chi^3+2S_2} \quad (20)$$

where we define the sums

$$S_1=\sum_{i=1}^{\infty}\frac{e^{-\mu_i^2x^*}}{\mu_i y_4\left(\mu_i,1\right)}\left[y_1\left(\mu_i,1\right)-9BrI_i\right] \quad (21)$$

$$S_2=\sum_{i=1}^{\infty}\frac{e^{-\mu_i^2x^*}}{\mu_i^3 y_4\left(\mu_i,1\right)}\left[y_1\left(\mu_i,1\right)-9BrI_i\right] \quad (22)$$

and the integral

$$I_1\equiv\int_0^1\chi^2y^{*2}\left(2ay^{*2}+1\right)y_2\left(\mu_i,y^*\right)dy^* \quad (23)$$

The numerical integration of Eq. (23) was performed by Romberg's integration procedu-re applied on the extended trapezoidal method.

For engineering purposes it is more advantageous to use the average Nusselt number from the inlet, derived from an energy balance and here referred to as $\overline{Nu}(x^*)$. This implicit equation must be solved numerically.

$$x^*\overline{Nu}=4\ln\left|\frac{-12Br\chi-\overline{Nu}}{\left(\overline{\theta}-1\right)\overline{Nu}-12Br\chi}\right| \quad (24)$$

Discussion

It is convenient to separate the analysis of the results of channel heating (*Br*>0) and cooling (*Br*<0).

Fluid heating (Br > 0)

This situation is represented in Fig. 1 and corresponds to $T_w>T_i$. The direction of heat transfer at the wall varies along the pipe when viscous dissipation is accounted for.

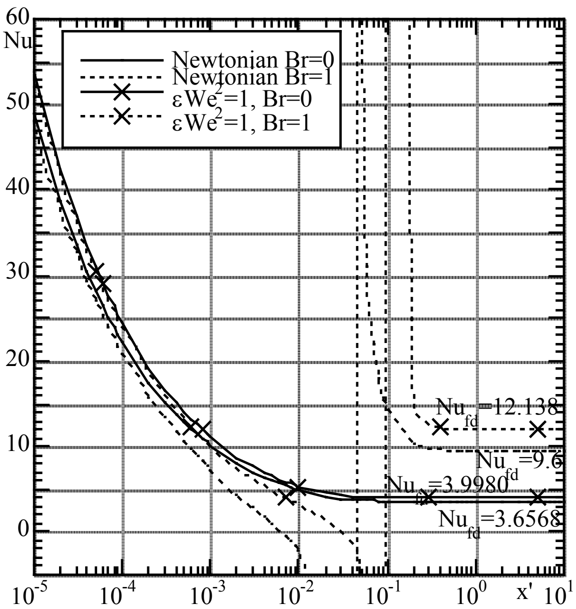

Figure 1- Effects of viscous dissipation and elasticity/extensibility on the Nusselt number variation for fluid heating (Br>0) .

Initially, (low *x*', where $x'\equiv x/D_HRePr=x^*/16$), heat transfer is from the wall to the fluid and the Nusselt number is positive because $T_w>T_b$ ($Nu=D_H\dot{q}_w/k(T_w-T_b)$). Viscous dissipa-tion increases the fluid temperature especially near the wall where the velocity gradients are steeper. At a $x'=x_1'$ the wall radial temperature gradient vanishes ($(\partial T/dy)_w=0\Rightarrow\dot{q}_w=0$), but since $T_w>T_b$, then *Nu*=0 and becomes negative for $x'>x'_1$. Viscous dissipation keeps increasing T_b

so that at $x' = x'_2$, $T_b = T_w$ resulting in a singular *Nu*. The *Nu* vs. *x*' variation for Newtonian fluids was discussed by [10] and is also plotted in Fig. 1.

The effect of shear thinning (*a*≠0) is to increase *Nu* and to delay the above transitions (x'_1 and x'_2), as shown in Fig. 1.

With viscous dissipation, the asymptotic *Nu* is independent of *Br* and the values found correspond to those in the literature for Newtonian [4] and the PTT fluid [7].

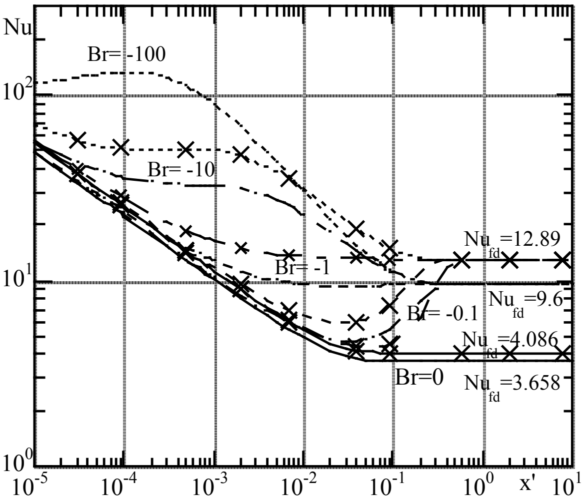

Figure 2- Effect of viscous dissipation and elasticity/extensibility on the Nusselt number variation for fluid cooling (Br<0). No symbols $\varepsilon We^2 = 0$, crosses for $\varepsilon We^2 = 10$.

Fluid cooling case (Br < 0)

Here, $T_w < T_i$ so *Br*< 0. The influence of *Br* is similar for the Newtonian and the PTT fluids, as shown in Fig. 2 (for εWe^2=0 and 10; *Br*=0,-0.1,-1,-10 and -100), and results from a balance between the opposite effects of wall cooling and heating by viscous dissipation. For Newtonian fluids there are two behaviours separated by the critical Brinkman number Br_1=-6/5 [4]. For $Br < Br_1$, viscous dissipation dominate and *Nu* decays monotonically with *x*' except at very low *x*' and high absolute *Br*. For $Br_1 < Br < 0$, the *Nu* vs. *x*' variation goes through a minimum at a certain critical axial position x'_c: for $x' < x'_c$, wall cooling is dominant and *Nu* decreases with *x*' whereas for $x' > x'_c$ viscous dissipation predominates and induces a raising tendency on the *Nu* vs. *x*' variation. At high *x*', when the thermal condition becomes fully-developed, *Nu* tends to a constant value independent of *Br*. Again, these asymptotic values of *Nu* coincide with those in the literature for the Newtonian [10] and the PTT cases [7].

Another feature seen in Fig. 2 regards the effect of viscoelastic/elongational fluid properties. Whereas under fully-developed flow *Nu* increases with εWe^2, during flow development the trend is reversed

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