Proc. EM2000 14th Engineering Mechanics Conference
ASCE, Ed. J. L. Tassoulas, Univ. Texas Austin, USA

EXTENSION OF A FINITE VOLUME METHOD IN SOLID STRESS ANALYSIS TO CATER FOR NON-LINEAR ELASTO-PLASTIC EFFECTS

C.J.Rente [a] , P.J.Oliveira [b]
[a] *Departamento de Engenharia Civil, Instituto Politécnico de Tomar, Estrada da Serra, 2300 Tomar, Portugal*
[b] *Departamento de Engenharia Electromecânica, Universidade da Beira Interior, Rua Marquês D'Ávila e Bolama, 6200 Covilhã, Portugal*
[a] c.rente@ipt.pt, [b] pjpo@ubi.pt

SUMMARY
This paper reports the development and application of a finite-volume based methodology which has been extended to cater for material non-linearities in the governing equations of solid stress analysis. The finite volume computational methodology uses indirect addressing to map multi-block geometries with complex shape, the governing equations are written in terms of general curvilinear co-ordinates but are solved for the cartesian components of the displacement vector, and the solution is obtained by sequential iteration through the linearised equation sets. The algebraic forms of the equations are easily obtained by integration over control volumes, thus ensuring a fully conservative scheme, and incremental stress-strain relationships are embedded into the sequential procedure. The novelty of the method lies on the use of generalised stress-strain constitutive relations for elastoplastic materials which are transformed into a form readily applicable in finite volume numerical analysis.

1. INTRODUCTION
Computational stress analysis in elastoplasticity with the Finite Element Method (FEM) is nowadays well established, usually involving an incremental-iterative procedure to gradually update the displacements, stresses and strains in order to accurately cater for material nonlinearities along the load (or time) axis [1,8]. In the past few years, the experience gained with the Finite Volume Method (FVM) in fluid flow simulations involving complex mathematical models with non-linear differential equations, typically leading to large matrices, has been imported to stress analysis (e.g. Demirdzic *et al.* [2-3]) and shows promising perspectives as a viable alternative solver to FEM in elastic-plastic solid body stress analysis.

In this paper the generalised incremental stress-strain relations for elastic-plastic materials are adapted for a FV formulation. This formulation is based on integral forms of the governing equations, with a second order accurate spatial discretization and a segregated solution procedure for solving iteratively the sets of resulting algebraic equations. Convergence of the iterative procedure is assured by a discretization practice leading to diagonally dominant matrices

2. GOVERNING EQUATIONS
The equations to be solved are those governing the behaviour of three-dimensional elastic-plastic solids. In what follows, tensorial notation with implied summation for repeated indices will apply to either Cartesian x_i $(i,j,...)$ or curvilinear ξ_l $(l,m,...)$ directions, as shown in Fig.1.

The momentum conservation equation, expressing Newton's second law for an

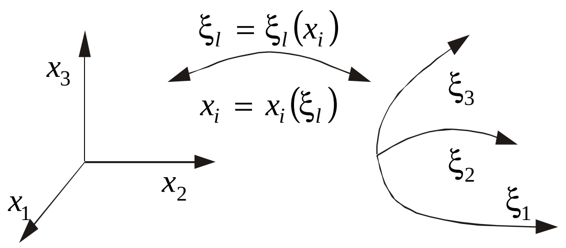

Fig.1. Scheme of transformation from rectangular Cartesian to a general frame..

arbitrary portion of solid with volume V, is written in differential form, referred to a rectangular Cartesian system, as:

$$\frac{\partial^2(\rho u_i)}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i \tag{1}$$

In (1), ρ is the solid density, u_i are the displacement vector components, σ_{ij} are the stress tensor components and g_i are body force components. The applications considered in this work are static and the time dependent term in (1) is used as an aid in the iterative-like time advancement. The elastic-plastic stress-strain relations for a general hardening plasticity model can be expressed in an incremental form as

$$d\sigma_{ij} = C_{ijkl}(d\varepsilon_{kl} - d\varepsilon_{kl}^p) \tag{2}$$

where C_{ijkl} is the isotropic elastic response, $d\varepsilon_{kl}$ is the total strain increment and $d\varepsilon_{kl}^p$ is the plastic strain increment. The 4th order tensor C_{ijkl} is expressed as:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{3}$$

where the Lamé's material constants are λ and μ and the Kronecker symbol is δ_{ij}. The plastic strain increment are given by [6]:

$$d\varepsilon_{ij}^p = \frac{\dfrac{\partial f}{\partial \sigma_{kl}} C_{klmn} d\varepsilon_{mn}}{H' + \dfrac{\partial f}{\partial \sigma_{ab}} C_{abcd} \dfrac{\partial g}{\partial \sigma_{cd}}} \frac{\partial g}{\partial \sigma_{ij}} \tag{4}$$

where f and g are respectively the yield function and the plastic potential function and H' is the hardening modulus. Substituting (4) into (2), we find the general form of the elastic-plastic constitutive equations, valid for perfect plastic, isotropic strain-hardening, and anisotropic strain-hardening materials:

$$d\sigma_{ij} = \left[C_{ijkl} - \frac{\dfrac{\partial g}{\partial \sigma_{rs}} C_{ijrs} C_{mnkl} \dfrac{\partial f}{\partial \sigma_{mn}}}{H' + \dfrac{\partial f}{\partial \sigma_{ab}} C_{abcd} \dfrac{\partial g}{\partial \sigma_{cd}}} \right] d\varepsilon_{kl} \tag{5}$$

Restricting ourselves to isotropic hardening under the assumption of the associated flow rule ($f \equiv g$) and recalling that the yield functions can be expressed in terms of the stress invariants $I_1 = \sigma_{ii}$, $J_2 = s_{ij}s_{ij}/2$ and $J_3 = s_{ij}s_{jk}s_{ki}/3$, the derivatives $\partial f / \partial \sigma_{ij}$ in (5) can generally be written as

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial I_1}\frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial f}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial f}{\partial J_3}\frac{\partial J_3}{\partial \sigma_{ij}} = A\ \delta_{ij} + B\ s_{ij} + C\ t_{ij} \tag{6}$$

with $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$ and $t_{ij} = s_{ik}s_{kj} - 2J_2\delta_{ij}/3$.

After inserting (3) and (6) into (5) followed by some algebraic manipulations, we obtain the incremental stress strain relations $d\sigma_{ij} = C^{ep}_{ijkl}d\varepsilon_{kl}$ where C^{ep}_{ijkl} is the elastic-plastic tensor

$$C^{ep}_{ijkl} = C_{ijkl} - \frac{1}{H}H_{ij}H_{kl} \tag{7}$$

with

$$H = 3A^2(3\lambda + 2\mu) + 4\mu B^2 J_2 + 12\mu BCJ_3 + 2\mu C^2(s_{ik}s_{kj}s_{il}s_{lj} - \frac{4}{3}J_2^2) + H' \tag{8}$$

$$H_{ij} = A(3\lambda + 2\mu)\delta_{ij} + 2\mu B s_{ij} + 2\mu C t_{ij} \tag{9}$$

Finally, the incremental stress-strain relation can be presented in the form

$$d\sigma_{ij} = 2\mu d\varepsilon_{ij} + \lambda\delta_{ij}d\varepsilon_{kk} - \frac{1}{H}H_{ij}H_{kl}d\varepsilon_{kl} \tag{10}$$

with the incremental strain tensor defined as

$$d\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i}\right) \tag{11}$$

After introducing the constitutive equation (10) into the momentum balance equation (1), this can be rewritten as

$$\begin{aligned} &\frac{\partial^2 \delta u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\left(\mu\frac{\partial \delta u_i}{\partial x_j}\right) = \\ &\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \delta u_j}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(\lambda\frac{\partial \delta u_k}{\partial x_k}\right) - \frac{\partial}{\partial x_j}\left(\frac{H_{ij}H_{kl}}{H}\frac{\partial \delta u_k}{\partial x_l}\right) + \rho g_i \end{aligned} \tag{12}$$

representing a set of three partial differential equations , subject to appropriate initial and boundary conditions, to be solved for the displacements increments aligned with the Cartesian co-ordinates. With the displacements increments obtained from (12), the increments of stress and strains components can easily be calculated.

3. NUMERICAL METHOD

The computational domain is divided into contiguous six-faced cells with arbitrary orientation, and the differential equations arising from the mathematical model are then integrated over each cell [5]. The various terms in the equations are discretized by means of central differences in the mesh formed by those cells, with the variables stored at the centre of the cells. The end result of the discretization is 3 sets of algebraic equations relating centre of cell values of the unknown variables to their neighbouring cell values

The integration of the governing equations written in generalised co-ordinates is straightforward after an acquaintance with the nomenclature introduced in Fig.2. First the Cartesian form of the equations is transformed into generalized co-ordinates by application of the rule [5]:

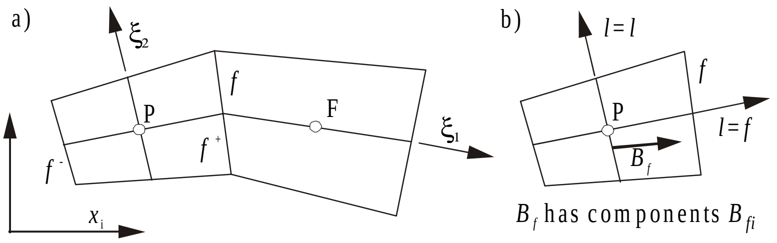

Fig.2. Some nomenclature: a) generic and neighbouring cells; b) area vectors and components

$$\frac{\partial}{\partial x_i}=\frac{\partial}{\partial \xi_l}\frac{\partial \xi_l}{\partial x_i}=\frac{1}{J}\frac{\partial}{\partial \xi_l}\beta_{li} \tag{13}$$

where J is the Jacobian of the transformation $x_i = x_i(\xi_l)$ and β_{li} are metric coefficients defined as the cofactor of $\partial x_i/\partial \xi_l$. Then, integration is equivalent to replacing: the β_{li} coefficients by the Cartesian i-components of the surface along direction *l*, denoted B_{li}; the Jacobian J by the cell volume V; and the derivatives $\partial/\partial \xi_l$ by differences between values along direction *l*.

Application of the transformation rule (13) to momentum balance equation (1) leads to the final form of the governing equations:

$$\frac{\partial^2 (J\rho \delta u_i)}{\partial t^2}=\frac{\partial}{\partial \xi_l}\left(\beta_{lj} d\sigma_{ij}\right)+J\rho g_i \tag{14}$$

Integration over a cell V_P of each term in this equation was outlined in our previous work [5] and so here we give details only for the stress divergence term:

$$\int_{V_P}\frac{\partial}{\partial \xi_l}\left(\beta_{lj} d\sigma_{ij}\right)\mathrm{d}V=\sum_f\left(B_f T_{i,f}\right) \tag{15}$$

where *f* denotes the faces of the general cell P, $\boldsymbol{T}$ is the traction vector and B_f is the surface area of cell face *f*. Eq. (15) thus represents the sum of the forces externally acting on cell P. The force applied at cell face *f* is, from (10) , (12) and (13):

$$\begin{aligned} B_f T_{i,f} &= \frac{\mu_f}{V_f}\left(B_{fj}B_{lj}\frac{\partial \delta u_i}{\partial \xi_l}+B_{fj}B_{li}\frac{\partial \delta u_j}{\partial \xi_l}\right)_f \\ &+\frac{\lambda_f}{V_f}\left(B_{fi}B_{lk}\frac{\partial \delta u_k}{\partial \xi_l}\right)_f-\left\{\frac{1}{V_f}\left(\frac{H_{ij}H_{kr}}{H}B_{fj}B_{lr}\frac{\partial \delta u_k}{\partial \xi_l}\right)\right\}_f \end{aligned} \tag{16}$$

where the term in curled brackets represents the contribution from elasto-plasticity and is activated in cells where the effective stress (from the adopted yield criterion) is greater than the yield stress,Y

The final form of the momentum balance is obtained after re-grouping the various terms to give linearized sets of equations:

$$a_P\, \delta u_{i,P}-\sum_F^6\left(a_F\, \delta u_{i,F}\right)=S_i \tag{17}$$

to be solved for the unknown displacements $\delta u_{i,P}$. The terms written on the lhs of (17) are treated implicitly and the coefficients a_P and a_F of the matrix are given by (see [5]):

$$a_P = \sum a_F + a_{inertia} \qquad a_F = \left(2\mu_f + \lambda_f\right) B_f^2 \big/ V_f \tag{18}$$

where $a_{inertia}$ represents the contribution of the inertia term. The source term S_i is treated explicitly and groups terms like those in (16) that are not included in the coefficients. The resulting system matrix (17) is symmetric positive-definite and well-structured (with only seven nonzero diagonals in 3-D) allowing solution by means of an efficient symmetric conjugate-gradient solver, pre-conditioned with incomplete Cholesky decomposition [4].

4. TEST CASE

The present finite volume methodology is demonstrated in this section on a plane stress problem of a perforated plate, with and without strain hardening. A finite plate with a circular hole in its centre is loaded by a unidirectional uniform tension, as in Fig.3 a).

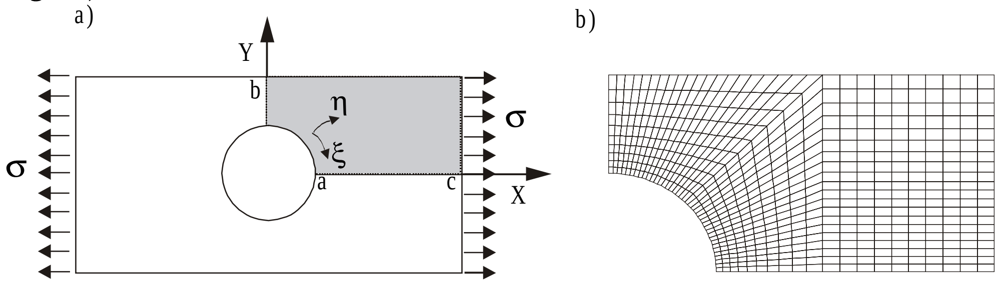

Fig 3. Plate with a central circular hole subjected to unidirectional tensile load : (a) geometry and dimensions , (b) numerical grid. (E=68.7 GPa,ν =0.3, Y=238 MPa, H =2.21 GPa)

By taking into account the symmetry of the problem, the analysis was performed on only one quarter of the solution domain in a mesh with 600 cells with a = b/2 = 5 mm and c = 18 mm as shown in Fig.3 b). This problem was investigated experimentally by Theocaris and Marketos [7] whose results, relating the maximum longitudinal strain to the applied stress, are plotted in Fig.4 together with the present predictions (lines). The Von Mises criterion is applied in two situations: for ideal plasticity (plastic modulus

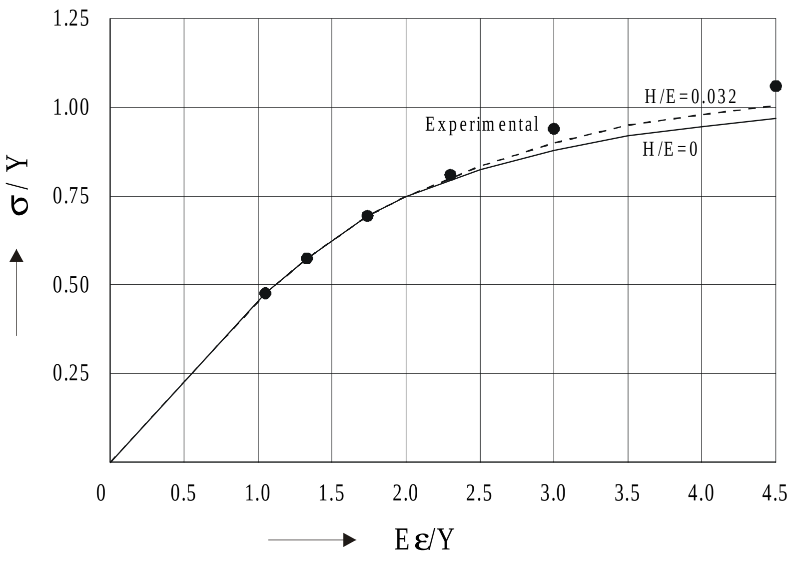

Fig.4. Relation between applied stress and maximum longitudinal strain in a perforated plate.

H=0) and for strain hardening, corresponding to a constant plastic modulus (H/E=0.032). The predictions are in good agreement with the experimentat results especially for strain hardening.

For the situation corresponding to ideal plasticity the spread of the plastic zone at various load levels (applied tension over initial yield stress) is presented in Fig.5. For higher loading the plastic zone spreads further away, above the hole, where stresses are expected to be more intense. The contours in Fig. 5 compare favourably with the solution given in [8], in spite of the relative coarse mesh used in this demonstration. Mesh refinement in zones of high stress gradients is required to improve the predictions and this will be carried out in future work

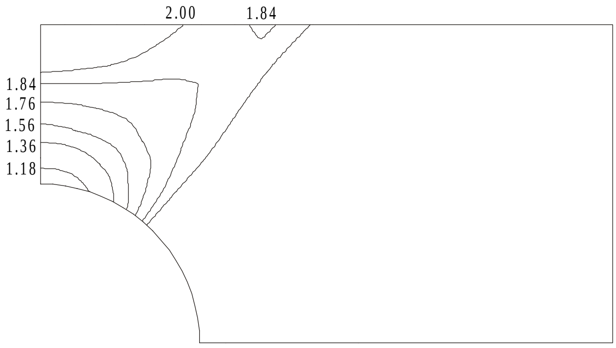

Fig.5. Development of plastic zones (parameter is σ / Y).

5. CONCLUSIONS

A generalized stress-strain constitutive equation adequate for elastoplastic materials is implemented in a finite volume method which is applied to the solution of a perforated plate in plane stress. Comparison with available results shows that the method is able to provide predictions with good accuracy. Compared with standard FEM, the memory requirements are much smaller and thus larger problems may be considered. In terms of solution efficiency, no conclusions could be drawn because no attempt has been made to include required acceleration techniques. This is left for future work.

6. REFERENCES

[1] BATHE, K - Finite Element Procedures in Engineering Analysis, Prentice-Hall, Inc., New Jersey, 1982.

[2] DEMIRDZIC, I and MARTINOVIC, D - Finite Volume Method for Thermo-Elasto-Plastic Stress Analysis, Comput Meth. Appl. Mech. Engrg., 109, pp. 331-349, 1993.

[3] DEMIRDZIC, I, MUZAFERIJA, S and PERIC, M - Advances in Computation of Heat Transfer, Fluid Flow and Solid Body Deformation Using Finite Volume Approaches, Advances in Numerical Heat Transfer, Ed. Minkowycz, and Sparrow, 1997.

[4] MEIJERINK, JA and VAN DER VORST, HA - An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix, Math. of Comp., 31, pp. 148-162, 1977.

[5] OLIVEIRA,PJ and RENTE,CJ – Development and Application of a Finite Volume Method for Static and Transient Stress Analysis, Proc.of NAFEMS World Congress'99 on Effective Engineering Analysis, Vol.1, pp.297-309,Newport, Rhode Island, USA,1999.

[6] CHEN,WF and MIZUNO,E - Nonlinear Analysis in Soil Mechanics: Theory and Implementation, Elsevier, 1990.

[7] THEOCARIS,PS and MARKETOS,E – J. Mech. Phys. Solids, 12, 377, 1964.

[8] ZIENKIEWICZ,OC and TAYLOR,RL – Finite Element Method – Solid and Fluid Mechanics: Dynamics and Nonlinearity, Vol.2, McGraw-Hill Publ, New York, 1991.