JUSTIFICATION FOR USING A VARIABLE DEBORAH NUMBER FUNCTION OF THE SHEAR-RATE TO CHARACTERISE VISCOELASTIC FLOWS

P.J. Oliveira

Departamento de Engenharia Electromecânica

Universidade da Beira Interior, 6200 Covilhã.

Resumo

Muitos autores em trabalho experimental com fluidos viscoelásticos fazem a caracterização do escoamento utilizando um número adimensional de Deborah (ou de Weissenberg) que varia em função de uma taxa de deformação $\dot{\gamma}$. A razão para esta escolha é, segundo esses autores, de reflectir nos parâmetros adimensionais o efeito de "*shear-thinning*" do fluido mas, se análise dimensional for aplicada às equações que regem o escoamento e os fluidos, vê-se que os parâmetros resultantes são fixos (não dependem de $\dot{\gamma}$). Para se tentar justificar o uso de $De(\dot{\gamma})$ e $We(\dot{\gamma})$ foi analisada uma solução analítica em escoamento completamente desenvolvido num canal plano, com um fluido do tipo PTT. A análise mostra que a solução exacta pode ser de facto expressa em termos de $De(\dot{\gamma}_{\text{caracter.}})$ fornecendo uma justificação para a prática utilizada.

Abstract

In experimental studies involving flows of viscoelastic liquids it is customary to characterise the observed behaviour with a variable Deborah number, $De(\dot{\gamma})$ function of a typical shear-rate $\dot{\gamma}$, instead of a naturally-defined fixed Deborah number, De_o. Since this choice does not arise naturally from application of dimensional analysis to the governing equations, we seek a justification from analytical solutions for simpler flows. In this work we take a fully-developed duct flow for which an analytical solution is available, and by analysis we seek whether the solution can be expressed in terms of $De(\dot{\gamma})$ where $\dot{\gamma}$ is a characteristic shear rate in the flow. We find that the solution can indeed be expressed in terms of $De(\dot{\gamma}_{\text{charact.}})$.

Keywords: Stress scaling; shear-thinning; PTT model; Deborah $De(\dot{\gamma})$ and Weissenberg $We(\dot{\gamma})$ numbers.

Introduction

In experimental studies involving flows of viscoelastic liquids it is customary to characterise the observed behaviour with a variable Deborah number, $De(\dot{\gamma})$ function of a typical shear-rate $\dot{\gamma}$, instead of a naturally-defined fixed Deborah number, De_0. If we take the flow through abrupt contractions as an example, that characterisation choice is found in the work of McKinley et al. (1991), with a fluid exhibiting constant viscosity (η) but shear-thinning 1st-normal stress coefficient (Ψ_1), and in the work of Quinzani et al. (1994), when both η and Ψ_1 are shear-thinning.

Often these authors use differential constitutive equations with fixed parameters to represent their actual fluids, for example Quinzani et al. (1995) take the equation proposed by Phan-Thien and Tanner (1977) which in its simplified form with a linear stress coefficient is:

$$(1 + \frac{\epsilon \lambda_0}{\eta_0}\ \mathrm{tr}(\tau))\ \boldsymbol{\tau}\ +\ \lambda_0\ \overset{\triangledown}{\boldsymbol{\tau}}\ =\ 2\eta_0 \boldsymbol{D}. \tag{1}$$

Here, τ and D are the extra-stress and deformation-rate tensors, λ_0 is the relaxation time, η_0 is a constant viscosity coefficient (these two constants are often referred to as the zero-shear rate values) and $\overset{\triangledown}{\tau}$ denotes Oldroyd's upper convected derivative,

$$\overset{\triangledown}{\tau}\ =\ \mathrm{D}u/\mathrm{D}t\ -\ \boldsymbol{\tau} \cdot \nabla \boldsymbol{u}\ -\ \nabla \boldsymbol{u}^{\mathrm{t}} \cdot \boldsymbol{\tau}$$

If we take a velocity scale U, and a length scale L, then standard dimensional analysis applied to Eq. (1) leads to:

$$(1 + \epsilon De_0\ \mathrm{tr}(\boldsymbol{\tau}^*))\ \boldsymbol{\tau}^*\ +\ De_0\ \overset{\triangledown}{\boldsymbol{\tau}}^*\ =\ 2\boldsymbol{D}^* \tag{2}$$

where the asterisk denotes nondimensional variables ($\tau^* \equiv \tau/(\eta_0 U/L)$) and

$$De_0\ =\ \frac{\lambda_0 U}{L} \tag{3}$$

is the fixed Deborah number. It is then arguable why the relaxation time in the definition of the Deborah number should be a function of $\dot{\gamma}$, so that one ends up with the following definitions for the shear-rate dependent Deborah and Weissenberg numbers:

$$De(\dot{\gamma})\ =\ \frac{\lambda(\dot{\gamma})\,U}{L} \quad \text{and} \quad We(\dot{\gamma})\ = \lambda(\dot{\gamma})\dot{\gamma} \tag{4}$$

when in the constitutive equation we have a fixed value for the relaxation-time, λ_o.

In order to understand this question we take a simple flow, specifically a fully-developed duct flow for which an analytical solution is available for the PTT fluid (Oliveira and Pinho, 1999), and by analysis we seek whether the solution (found in in this last reference to be a function of De_o) can be expressed in terms of $De(\dot{\gamma}_{\text{charact.}})$. If that is the case in complex flows, like the abrupt contraction flows referred to above, then we expect the same to occur in this simpler flow.

The present analysis is valid only for the PTT constitutive equation. This model has often been used in viscoelastic fluid flow simulations and if we focus only in contraction flows, which bear a close analogy with the duct type flows here considered, we may mention the works of White and Baird (1988), Baaijens (1993), Carew et al. (1993), Azaiez et al. (1996) and Baloch et al (1996). All these authors have used forms of the PTT model and such widespread interest on this particular viscoelastic model gives support for the present effort.

Analysis for the Linear PTT model

The analysis is valid for two-dimensional channel and axisymmetric pipe flows, with the streamwise velocity component denoted by u, and y representing either a transversal or a radial coordinate. The centreline/axis is located at $y = 0$ and $y = L$ is the wall. L is the half channel width or the pipe radius, and U is the cross-section average velocity. Index j identifies the flow case, with $j = 0$ for the plane and $j = 1$ for the axisymmetric flow case.

We start by rewriting the relevant equations from the exact solution derived by Oliveira and Pinho (1999) for the linear PTT model, namely the velocity profile:

$$u^* \equiv \frac{u(y)}{U} = \kappa \frac{U_N}{U} \left(1 - (\frac{y}{L})^2\right) \left(1 + 4\kappa^2 \epsilon De_0^2 \, (\frac{U_N}{U})^2 (1 + (\frac{y}{L})^2)\right) \tag{5}$$

and the profiles for the two stress components:

$$\tau_{xx}^* \equiv \frac{\tau_{xx}}{2\kappa\eta_0 \frac{U}{L}} = 4\kappa De_0 \, (\frac{U_N}{U})^2 \, (\frac{y}{L})^2 \tag{6}$$

$$\tau_{xy}^* \equiv \frac{\tau_{xy}}{2\kappa\eta_0 \frac{U}{L}} = -\frac{U_N}{U} \frac{y}{L} \, . \tag{7}$$

The value of κ is either 1.5 for the channel or 2 for the pipe flow, and so the normalising stresses become equal to the wall shear-stress value for a Newtonian fluid (bringing a constant factor into the previous nondimensionalisation, cf. Eq 3). The parameter

$$\frac{U_N}{U} \equiv \frac{-(\mathrm{d}p/\mathrm{d}x)\, L^2}{2^{j+1}\kappa\eta_0 U}$$

in the above equations is a normalised pressure gradient and was shown to be given by:

$$\frac{U_N}{U} = \frac{(432)^{1/6} \, (\delta^{2/3} - 2^{2/3})}{6 \, b^{1/2} \, \delta^{1/3}} \tag{8}$$

with: $\alpha = 3^3 b + 4$, $\delta = \alpha^{1/2} + 3^{3/2} b^{1/2}$ and

$$b \equiv \frac{8(3+j)\kappa^2}{(5+j)} \epsilon \, De_0^2 \, . \tag{9}$$

The above solution is function of two dimensionless parameters, the extensional parameter of the PTT, ϵ, and the fixed-scale based Deborah number De_0 defined by Eq. (3). As mentioned in the Introduction, many authors base their nondimensional parameters on shear-rate dependent values with a Weissenberg number defined as in Eq. (4) evaluated at a characteristic shear rate. The purpose of this work is to see whether the above analytical solution, Eqs. (5-8), can be written in terms of a characteristic shear-rate dependent dimensionless parameter, thus providing a justification for the choice of those many authors.

A variable relaxation time is usually defined (Bird et al. 1987) from the first normal stress coefficient thus:

$$\lambda(\dot{\gamma}) = \frac{\Psi_1(\dot{\gamma})}{2\eta(\dot{\gamma})}$$

or, since $\Psi_1 \equiv N_1/\dot{\gamma}^2$ and in the present situation the only nonvanishing normal stress is τ_{xx}, we may write:

$$\lambda(\dot{\gamma}) = \frac{\tau_{xx}(\dot{\gamma})}{2\dot{\gamma}^2\eta(\dot{\gamma})}$$

Upon substitution in Eq. (4) we get:

$$We(\dot{\gamma}) = \frac{\tau_{xx}(\dot{\gamma})}{2\dot{\gamma}\eta(\dot{\gamma})}. \tag{10}$$

Oliveira and Pinho also give the variation of the shear viscosity across the channel or pipe as

$$\eta(\dot{\gamma}) \equiv \frac{-\tau_{xy}}{\dot{\gamma}} = \frac{\eta_0}{\left(1 + 8\kappa^2 \, \epsilon \, De_0^2 (\frac{U_N}{U})^2 (\frac{y}{L})^2 \right)} \tag{11}$$

and this equation, together with Eqs. (6) for the normal stress component and (7) for the shear component, can be substituted into (10) to get:

$$We(\dot{\gamma}) = 2\kappa De_0 \, \frac{U_N}{U} \, \frac{y}{L}$$

In contrast to the previous work (Oliveira and Pinho) we find it more convenient to work here with a Weissenberg instead of a Deborah number. With κ taking the value of 1.5 in a

plane channel and 2 in a round pipe, we see that $2\kappa De_0$ has the meaning of a zero shear-rate Weissenberg number, that is

$$We_0 \ = \ 2\kappa De_0 \tag{12}$$

and so a scaled variation of the local Weissenberg number across the channel or pipe can be expressed as:

$$\frac{We(\dot{\gamma})}{We_0} \ = \ (\frac{U_N}{U})\,(\frac{y}{L})\,. \tag{13}$$

From this equation we see that at the centreline, $y = 0$ and $We(\dot{\gamma}) = 0$; at the wall, $y/L = 1$ and the wall value of the Weissenberg number becomes:

$$We_w \ \equiv We(\frac{y}{L} = 1) \ = \ We_0 \ \frac{U_N}{U} \tag{14}$$

Since in most works related to entry flows, or flows through contractions and expansions, the characteristic shear rate used to define the nondimensional parameters is that prevailing at the wall (typically where the shear is higher), then we seek now to express the solution in terms of We_w. After combining Eqs. (5), (9) and (14), and using the following expression given by Oliveira and Pinho:

$$\frac{U_N}{U} \ = \ \frac{1}{1+\,b\,(\frac{U_N}{U})^2} \tag{15}$$

we obtain the velocity profile as:

$$u^* \ \equiv \ \frac{u(y)}{U} \ = \ \kappa\,\left(1 - (\frac{y}{L})^2\right)\left(\frac{1 + \epsilon We_w^2\,(1+(\frac{y}{L})^2)}{1 + (\frac{2(3+j)}{5+j})\epsilon We_w^2}\right) \tag{16}$$

Similar manipulations for the stress components, from Eqs. (6) and (7), give:

$$\tau_{xx}^* \ \equiv \ \frac{\tau_{xx}}{2\kappa\eta_0\frac{U}{L}} \ = \ 2\,We_w\left(\frac{1}{1 + (\frac{2(3+j)}{5+j})\epsilon We_w^2}\right)(\frac{y}{L})^2 \tag{17}$$

$$\tau_{xy}^* \ \equiv \ \frac{\tau_{xy}}{2\kappa\eta_0\frac{U}{L}} \ = \ -\left(\frac{1}{1 + (\frac{2(3+j)}{5+j})\epsilon We_w^2}\right)(\frac{y}{L})\,. \tag{18}$$

The viscosity at the wall, a quantity with interest to the characterisation of a flow, is obtained from Eqs. (11) and (14) as

$$\frac{\eta_w}{\eta_0} = \frac{1}{(1+2\epsilon We_w^2)} \tag{19}$$

thus being also a function of We_w.

We therefore see that the velocity profile could be expressed in terms of a unique parameter ϵWe_w^2, where the wall value of the Weissenberg number is evaluated from $We_w = We(\dot{\gamma}_w)$ with the relationship between Weissenberg number and shear rate given by Eq. (13). Furthermore, the shear tress component also depends solely on ϵWe_w^2, whereas the normal stress component also depends separately on We_w alone.

Analysis for the Exponential PTT model

In this case the stress coefficient in the PTT equation takes an exponential variation:

$$\exp\left(\frac{\epsilon\lambda_0}{\eta_0}\,\mathrm{tr}(\boldsymbol{\tau})\right)\boldsymbol{\tau} + \lambda_0 \overset{\triangledown}{\boldsymbol{\tau}} = 2\eta_0 \boldsymbol{D}.$$

and a short analysis is sufficient to show that the same conclusions as above will apply without the necessity to derive the complete expressions. In Oliveira and Pinho it was shown that for this case the solution is a function of the parameter $b(U_N/U)^2$ with $b = 8\kappa^2\epsilon De_0^2$. Since Eq. (13) relating the wall Weissenberg number to U_N/U is still valid for this case (because Eqs. (6) and (7) are valid irrespective of the assumed stress coefficient), we see that the solution will be a function of:

$$b(U_N/U)^2 = 2\,\epsilon\,(2\kappa De_0 U_N/U)^2 = 2\,\epsilon\,We_w^2.$$

Therefore, we arrive at the same conclusions as above.

Final Remarks

The main results of the present analysis are embodied in Eqs. (16)-(18) which give the velocity and stress components profiles across a conduit (plane or axisymmetric) in terms of a characteristic Weissenberg number, We_w. A possible advantage of the present equations compared with the original solution of Oliveira and Pinho is that here the functional dependence of the solution on the 2 independent dimensionless parameters is explicit, whereas there it was buried into the U_N/U parameter.

If we link ϵ to the elongational characteristics of the fluid (recalling that the maximum uniaxial elongational viscosity decreases as ϵ increases) and We_w to elasticity, we see that u and τ_{xy} are only affected by a combination of the two through a single parameter $\epsilon^{1/2}We_w$ (which appears to the square). The normal stress, on the other hand, is also affected by the parameter We_w alone. One may then observe that τ_{xx} increases with We_w (a direct proportionality, characteristic of elasticity) but decreases with the combined parameter $\epsilon^{1/2}We_w$ (characteristic of "extensibility" of the fluid, or better extension-thinning).

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