Bifurcation phenomena in strong extensional flows
(in a cross-slot geometry)

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Outline

• Motivation
• Governing equations and numerical methods
• Results:
  – Weissenberg number $Wi_o$ at the stagnation point
  – Couette correction $C$
  – Asymmetry parameter $DQ$

• Discussion:
  – Stability
Motivation – Why the cross-slot?

- Strong extensional nature and free stagnation point
- Steady-state flow asymmetries at negligible Reynolds number
- Time-dependent flow for higher Deborah number


Earlier observations: Gardner et al., Polymer. 23 (1982) 1435–1442.
Motivation – Goals

• Accurate benchmark data:
  – Upper-Convected Maxwell (UCM) model
  – Oldroyd-B model
  – Simplified linear Phan-Thien-Than (sPTT) model

• Mechanism of Bifurcation:
  – Application of criteria developed to predict the onset of time-dependent instabilities
Governing equations

- Inertialess \((Re \to 0)\), isothermal, incompressible flow.

  - Conservation of mass: \(\nabla \cdot \mathbf{u} = 0\)

  - Conservation of momentum: 
    \[-\nabla p + \nabla \cdot \tau + \beta \eta_o \nabla^2 \mathbf{u} = 0\]
    \[
    \beta = \frac{\eta_s}{\eta_s + \eta_p}
    \]

  - Constitutive equation (sPTT model):
    \[
    \left(1 + \frac{\lambda \varepsilon}{(1 - \beta) \eta_o} \text{Tr}(\tau)\right) \tau + \lambda \left[\frac{\partial \tau}{\partial t} + \nabla \cdot \mathbf{u} \tau\right] = (1 - \beta) \eta_o \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) + \lambda \left(\tau \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot \tau\right)
    \]

  UCM or Oldroyd-B model for \(\varepsilon=0\)
Geometry
Numerical methods

• Fully implicit finite volume method (Oliveira et al., 1998)
  – Structured, collocated and non-orthogonal meshes
  – Time-marching algorithm
  – Diffusive terms: central differences scheme (CDS)

  – Advective terms, high resolution scheme: CUBISTA (Alves et al., 2003)

  – Log-conformation technique for polymeric stress tensor
    (Afonso et al., 2009)
Richardson extrapolation

- Given three meshes with cell spacing $h$, $2h$ and $4h$, for variable $\varphi$,

\[
\ln \left( \frac{\varphi_{2h} - \varphi_{4h}}{\varphi_{h} - \varphi_{2h}} \right) = \frac{2^p \varphi_{h} - \varphi_{2h}}{2^p - 1}
\]

order of convergence

extrapolated value
Meshes

<table>
<thead>
<tr>
<th>Mesh</th>
<th>NC</th>
<th>DOF</th>
<th>$\frac{\Delta x_{\text{min}}}{D}$</th>
<th>$\frac{\Delta y_{\text{min}}}{D}$</th>
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<tr>
<td>M1</td>
<td>12801</td>
<td>76806</td>
<td>0.02</td>
<td></td>
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<td>50601</td>
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<td>1207206</td>
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</table>
Bifurcation – UCM model
Streamlines superimposed onto contour plots of $N_1$

$D_e = 0.31$
Steady symmetry

$D_e = 0.315$
Steady asymmetry

$D_e = 0.33$
Prior to unsteady flow

$N_1 / (\eta U / D)$

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Bifurcation – Oldroyd-B model ($\beta=1/9$)
Streamlines superimposed onto contour plots of $N_1$

$De=0.36$
Steady symmetry

$De=0.37$
Steady asymmetry

$De=0.42$
Prior to unsteady flow
Bifurcation – sPTT model ($\beta=1/9$, $\varepsilon=0.02$)
Streamlines superimposed onto contour plots of $N_1$

$De=0.5$
Steady symmetry

$De=0.51$
Steady asymmetry

$De=0.9$
Prior to unsteady flow

$N_1/(\eta_0 U/D)$
Mesh-wise convergence
Demonstration with $Wi_o$

$Wi_o = \lambda \dot{\epsilon}_o$

sPTT model ($\beta = 1/9, \varepsilon = 0.02$)
Results

Weissenberg number $Wi_o$ at the stagnation point

$Wi_o \downarrow$

$\Rightarrow$ polymer relaxes

$\Rightarrow$ stress is relieved

(Oliveira et al., 2009)

No bifurcation without sufficiently high stress

(Xi and Graham, 2009)

(Afonso et al., 2010)


Afonso et al., J Non-Newton Fluid. 165 (2010) 743–751
Results – Couette Correction

Bifurcation

$C \downarrow$

$\Rightarrow$ Lower energy dissipation

(Poole et al., 2007)
(Oliveira et al., 2009)
(Afonso et al., 2010)

$C \equiv \frac{\Delta p - \Delta p_{fd}}{2\tau_w}$

Results – Asymmetry Parameter $DQ$

\[ DQ = \frac{q_2 - q_1}{q_1 + q_2} \]

Supercritical Pitchfork Bifurcation

Locally, \[ DQ = A \sqrt{De - De_{CR}} \]

\[ A = A(\varepsilon) \]

\[ De_{CR} = De_{CR}(\beta, \varepsilon) \]

(Rocha et al., 2009)

Discussion – Stability

- Effects of bifurcation
  - Stress is relieved
  - Pressure drop decreases

- => Greater stability

- Confirmation: analysis of bifurcation using instability criteria
  - M number (McKinley et al., 1996)
  - K number (Dou and Phan-Thien, 2008)

Discussion – M number

\[ M = \left[ \frac{\lambda U}{R} \frac{\tau_{ss}}{\eta_o \dot{\gamma}} \right]^{1/2} \]

Radial stress is generated if polymer molecules are displaced by disturbances.

(Pakdel and McKinley, 1996)

sPTT
\[ \beta = \frac{1}{9} \]
\[ \varepsilon = 0.02 \]

\[ {\text{De}} = 0.3 \]
\[ {\text{De}} = 0.5 \]


**M number definition**: McKinley et al., J Non-Newton Fluid. 67 (1996) 19-47.

**First description of mechanism**: Larson et al., J Fluid Mech 218 (1990) 573-600.

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Discussion – M number

sPTT, $\beta=1/9$, $\varepsilon=0.02$

According to M, no unstable regions near the stagnation point or birefringence strand => M number fails to predict bifurcation
Discussion – K number

\[ K = \left| \frac{\partial E}{\partial n} \right| \approx \left| \frac{\partial p}{\partial n} \right| = \left| \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \right| \]

Disturbances are amplified if the transverse energy gradient is large relative to the streamwise energy gradient.

\[ sPTT, \beta = 1/9, \varepsilon = 0.02 \]

\[ De = 0.3 \quad De = 0.5 \]


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Discussion – K number

$sPTT, \beta=1/9, \varepsilon=0.02$

According to K, the birefringence strand tends to instability only after bifurcation

$\Rightarrow$ K number fails to predict bifurcation
Conclusion

• Bifurcation improves stability.
  – Stress is relieved, pressure drop is minimized.

• Instability criteria fail to predict bifurcation.
  – Both the M and K numbers provide no indication that bifurcation is about to occur.

• => The two known types of elastic instability, steady bifurcation and time-dependent flow, apparently do not share a common cause.
Acknowledgements

European Research Council
Established by the European Commission
(Grant Agreement n. 307499)

Thank you for your attention!