IWNMNNF 2007
JUNE 6-10 RHODES

OSCILLATING CHANNEL FLOWS OF UCM AND OLDROYD-B FLUIDS: NUMERICAL AND ANALYTICAL SOLUTIONS

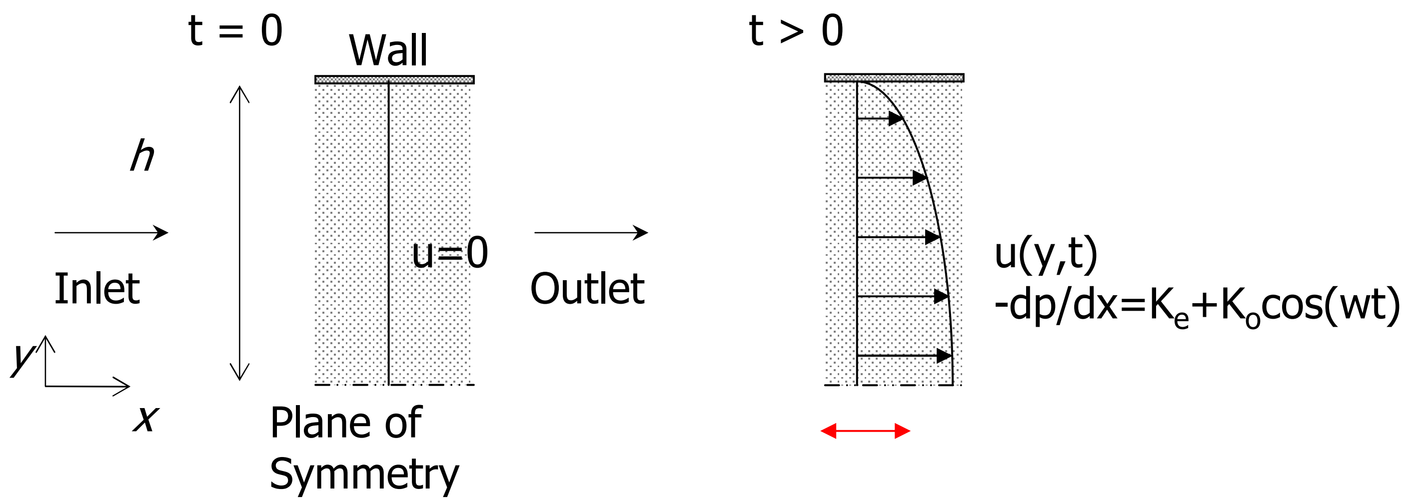
A.S.R. Duarte
A. I. P. Miranda
P. J. Oliveira

Universidade da Beira Interior

Dep. Eng. Electromecânica
UBI

MOTIVATION FOR OSCILLATING UCM and OLD-B CHANNEL FLOW

- "Simple" 1D unsteady flow
- Analytical solution can be derived
- Very convenient to assess and develop methods
- Relevance to blood flow
- More (and less) complicate than "startup flow"



PREVIOUS WORK

Oscillating pressure gradient (<u>Analytical</u>)

Pipe Flow

- **more work**
- **solution more complicate (Bessel functions)**
- **Bird et al. (Transport Phenomena,1960) (UCM)**
- **Rahaman & Ramkisson, JNNFM (1995) (UCM, OB)**

Channel Flow

- **much less work**
- **Newtonian: books (eg. Panton; White)**
- **UCM: Hayat et al., IJES (2001) (only velocity)**

PREVIOUS WORK

Simple unsteady flows (Numerical)

Solution: Waters & King

Startup of UCM and O-B

- Sato & Richardson, JNNFM (1994)
- Tanner & Xue, KARJ (2002); MIT (2003)
- Fietier & Deville, JCP (2003)
- Webster, Tamaddon & Aboubacar, JNNFM (2004)
- Xue, Tanner & Phan-Thien, JNNFM (2004)
- Ellero & Tanner, JNNFM (2005) (SPH sim.)

Pulsating Flow

- no work
- curved pipe: Manos, Marinakis & Tsangaris, JNNFM (2006)

DIFFERENCE WITH “STARTUP” FLOW

- “Startup” is transient; “Oscillating” is periodic (fully established)
- “Startup” more dependent on exact initial conditions
- “Startup” generates “shocks” leading to discontinuities
- Evaluating errors is more problematic (2nd order becomes 1st order)
- However, as will be shown, UCM oscillating requires very fine grids
- Both are adequate for numerical assessment (analytical solutions)

GENERAL EQUATIONS

Mass

$$\nabla \cdot \boldsymbol{u} = 0$$

Motion

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nabla \cdot \boldsymbol{\tau}$$

Constitutive (for the stress tensor τ)

$$\lambda_1 \overset{\nabla}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\eta_0 \left(\mathbf{D} + \lambda_2 \overset{\nabla}{\mathbf{D}} \right)$$

$$\boxed{\beta = \lambda_2 / \lambda_1}$$

Oldroyd derivative: $\overset{\nabla}{\boldsymbol{\tau}} = \frac{D\boldsymbol{\tau}}{Dt} - (\boldsymbol{\tau} \cdot \nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{\tau})$

ANALYTICAL SOLUTION (1) Equations for fully developed flow

Starting equations

$$\begin{cases} \rho \dfrac{\partial u}{\partial t} = \dfrac{\partial \tau_{xy}}{\partial y} - \dfrac{dp}{dx} \\ \tau_{xy} + \lambda_1 \dfrac{\partial \tau_{xy}}{\partial t} = \eta_0 \dfrac{\partial u}{\partial y} + \lambda_2 \eta_0 \dfrac{\partial^2 u}{\partial y \partial t} \end{cases}$$

Gives:

$$\rho\left(1+\lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} = \eta_0\left(1+\lambda_2 \frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2} - \left(1+\lambda_1 \frac{\partial}{\partial t}\right)\frac{dp}{dx}$$

Pressure gradient

$$\boxed{-\frac{1}{\rho}\frac{dp}{dx} = K_e + K_o \cos(\omega t)}$$

$$\boxed{u(y,t) = u_e(y) + u_o(y,t)}$$

Velocity decomposition

ANALYTICAL SOLUTION (2) Velocity solution, steady plus oscillating

Scaling: $U = u\dfrac{\omega}{K_o}$ $\quad Y = \dfrac{y}{h}$ $\quad T = \omega t$

Steady solution

$$U_e(Y) = \frac{\alpha^2}{2}\frac{K_e}{K_o}\left(1 - Y^2\right)$$

Eq. for osc. vel:

$$\left[1 + \omega\lambda_1 \frac{\partial}{\partial T}\right]\frac{\partial U_o}{\partial T} = \left[1 + \omega\lambda_1 \frac{\partial}{\partial T}\right]\cos(T) + \frac{1}{\alpha^2}\left[1 + \omega\lambda_2 \frac{\partial}{\partial T}\right]\frac{\partial^2 U_o}{\partial Y^2}$$

Separation of variables:

$$U_o(Y,T) = \mathrm{Re}\left[F(Y)\exp(iT)\right] \quad \Rightarrow iz_1 F = z_1 + \frac{z_2}{\alpha^2}F''$$

Oscillating solution

$$U_o(Y,T) = \mathrm{Re}\left\{ i\left[\frac{\cosh(zY)}{\cosh(z)} - 1\right]\exp(iT)\right\}$$

$$z^2 = \alpha^2 i \frac{z_1}{z_2}$$

ANALYTICAL SOLUTION (3) Velocity profile and controlling parameters

$$U(Y,T)=\frac{\alpha^2}{2}\frac{K_e}{K_o}\left(1-Y^2\right)+\mathrm{Re}\left\{i\left[\frac{\cosh(zY)}{\cosh(z)}-1\right]\exp(iT)\right\}$$

Non-dimensional groups:

$$z^2=\alpha^2 i\frac{1+i\omega\lambda_1}{1+i\omega\lambda_2}=\alpha^2 i\frac{1+i\alpha^2 E}{1+i\beta\alpha^2 E}$$

Womersley number: $$\alpha^2=\frac{\rho h^2\omega}{\eta_0}$$

Elasticity number: $$E=\frac{\lambda_1\eta_0}{\rho h^2}$$

Pressure gradient ratio: $$\frac{K_o}{K_e}$$

ANALYTICAL SOLUTION (4) Equation for the elastic shear stress

Starting from momentum eq.

$$\rho\frac{\partial u}{\partial t}=\frac{\partial\tau_{xy}}{\partial y}+\eta_2\frac{\partial^2 u}{\partial y^2}-\frac{dp}{dx}$$

Integration:

$$\tau_{xy}=\rho\int_0^y\frac{\partial u}{\partial t}dy'-\rho\left(K_e+K_o\cos\omega t\right)y-\eta_2\frac{\partial u}{\partial y}$$

Using the solution for velocity:

$$\boxed{T_{xy}=-\frac{\alpha^2K_e}{K_0}(1-\beta)Y-\mathrm{Re}\left\{\exp(iT)\frac{\sinh(zY)}{\cosh(z)}\left[\frac{\alpha^2}{z}+i\beta z\right]\right\}}$$

ANALYTICAL SOLUTION (5) Equation for the elastic normal stress component

Old-B Eq. (elastic component):

$$\tau_{xx} + \lambda_1 \frac{\partial \tau_{xx}}{\partial t} = 2\lambda_1 \tau_{xy} \frac{\partial u}{\partial y}$$

1st integration:

$$\tau_{xx} = \exp\left(-t/\lambda_1\right) \int_{-\infty}^{t} 2\tau_{xy} \frac{\partial u}{\partial y} \exp\left(t'/\lambda_1\right) dt'$$

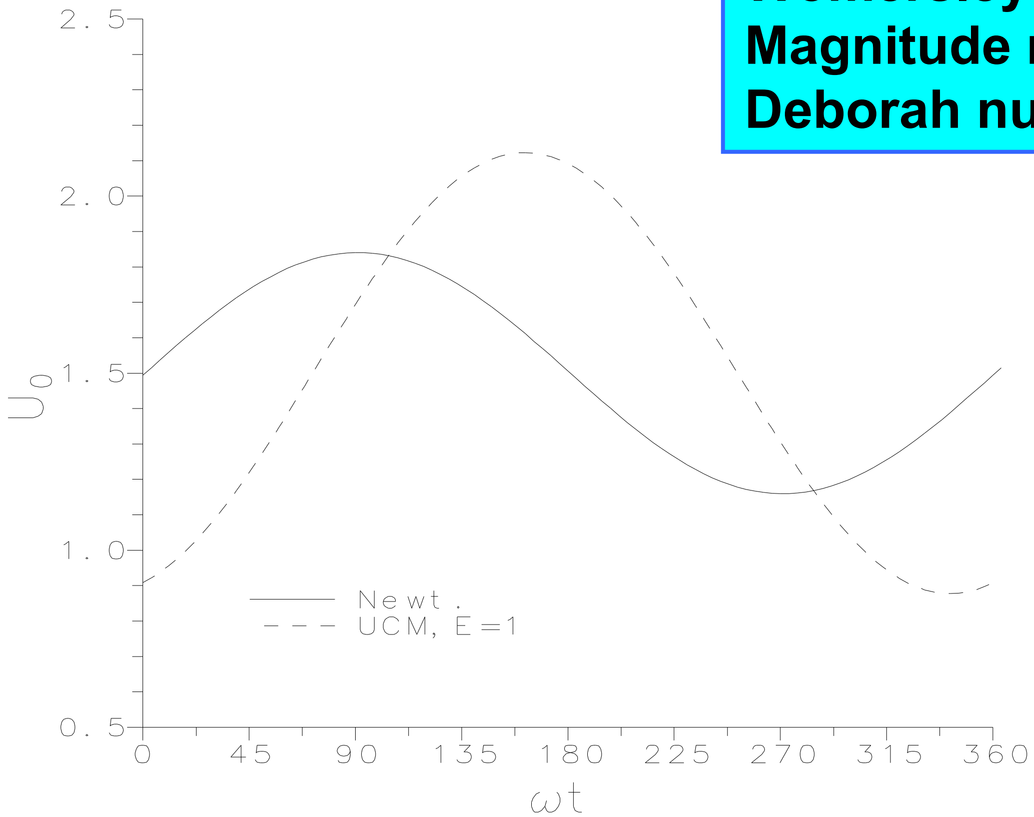
Nondimensional stress:

$$\begin{aligned} T_{xx} &= 6De \frac{\alpha^2 K_e}{K_0}\left(1-\beta\right)Y^2 \\ &- 6DeY \operatorname{Re}\left\{ \frac{\exp\left(iT\right)}{1+i\alpha^2 E} \frac{\sinh\left(zY\right)}{\cosh\left(z\right)} \left[iz\left(1-2\beta\right) - \frac{\alpha^2}{z} \right] \right\} \\ &- 6\frac{K_0}{K_e} De \operatorname{Re}\left\{ \frac{\exp\left(i2T\right)}{1+i2\alpha^2 E} \left(\frac{\sinh\left(zY\right)}{\cosh\left(z\right)} \right)^2 \left[i - \frac{\beta z^2}{\alpha^2} \right] \right\} \end{aligned}$$

$$De = \frac{\lambda_1 \bar{u}_e}{h}$$

UCM AND NEWTONIAN FLOWS: analytical centerline velocity

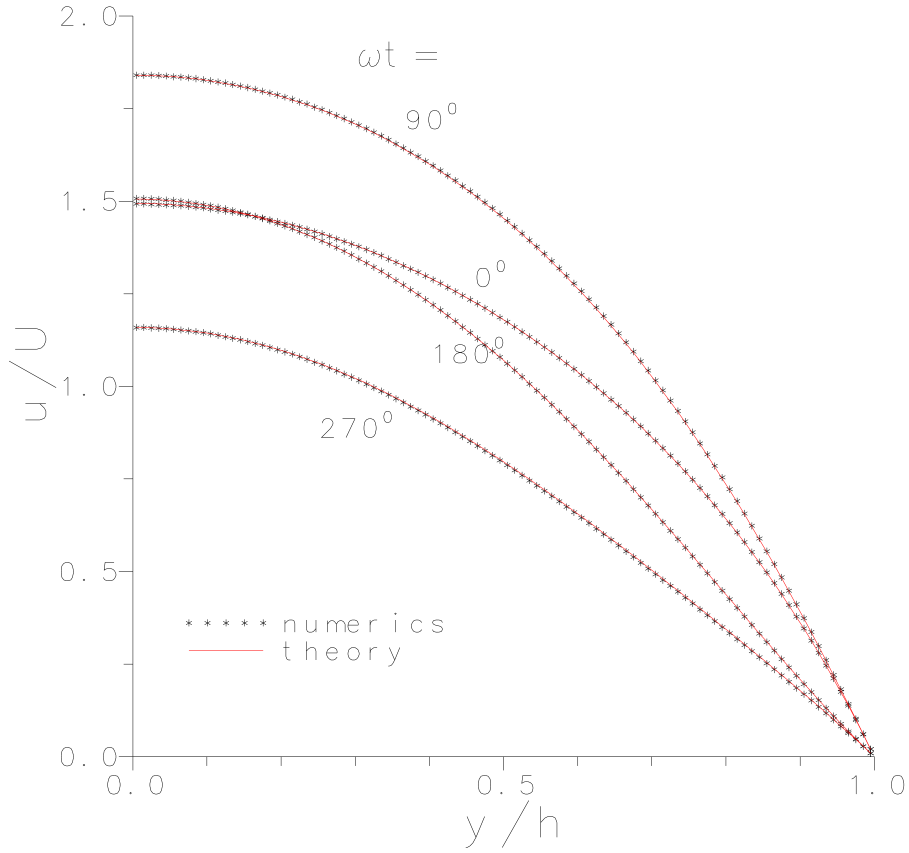
Elasticity number, E=1
Womersley number, α=4.9
Magnitude ratio, K_o/K_e=2.6
Deborah number, De=3.8



RESULTS FOR NEWTONIAN

$$\boxed{\beta = 1}$$

NEWTONIAN FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

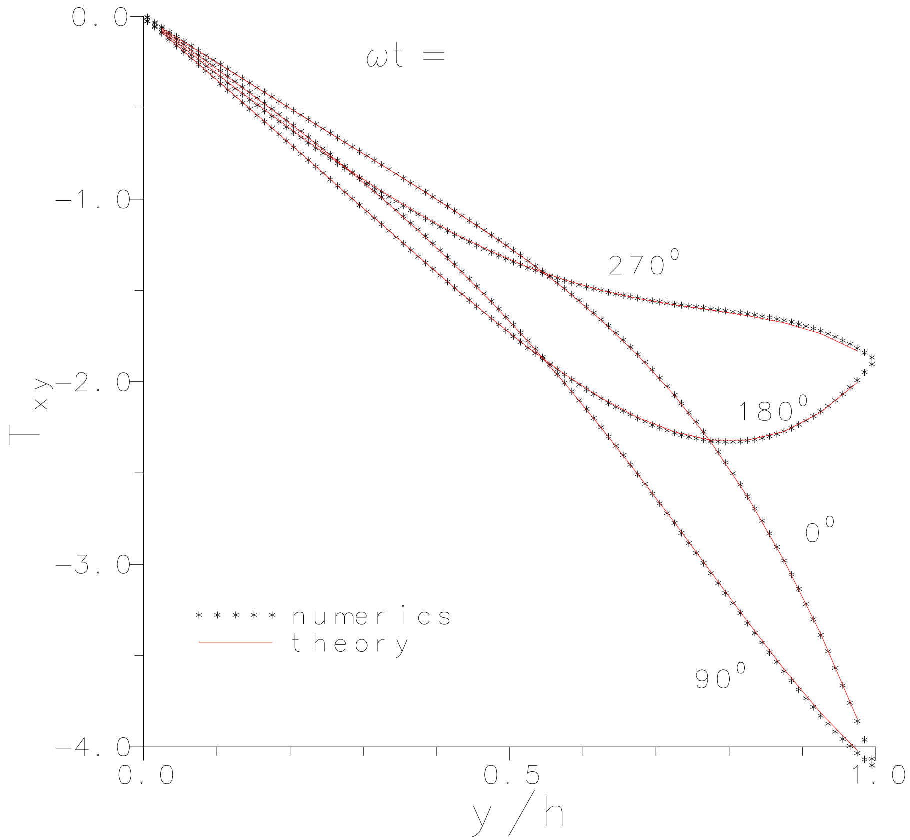


symbols- predictions

red lines- analytical solution

Velocity profiles

NEWTONIAN FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT



symbols- predictions

red lines- analytical solution

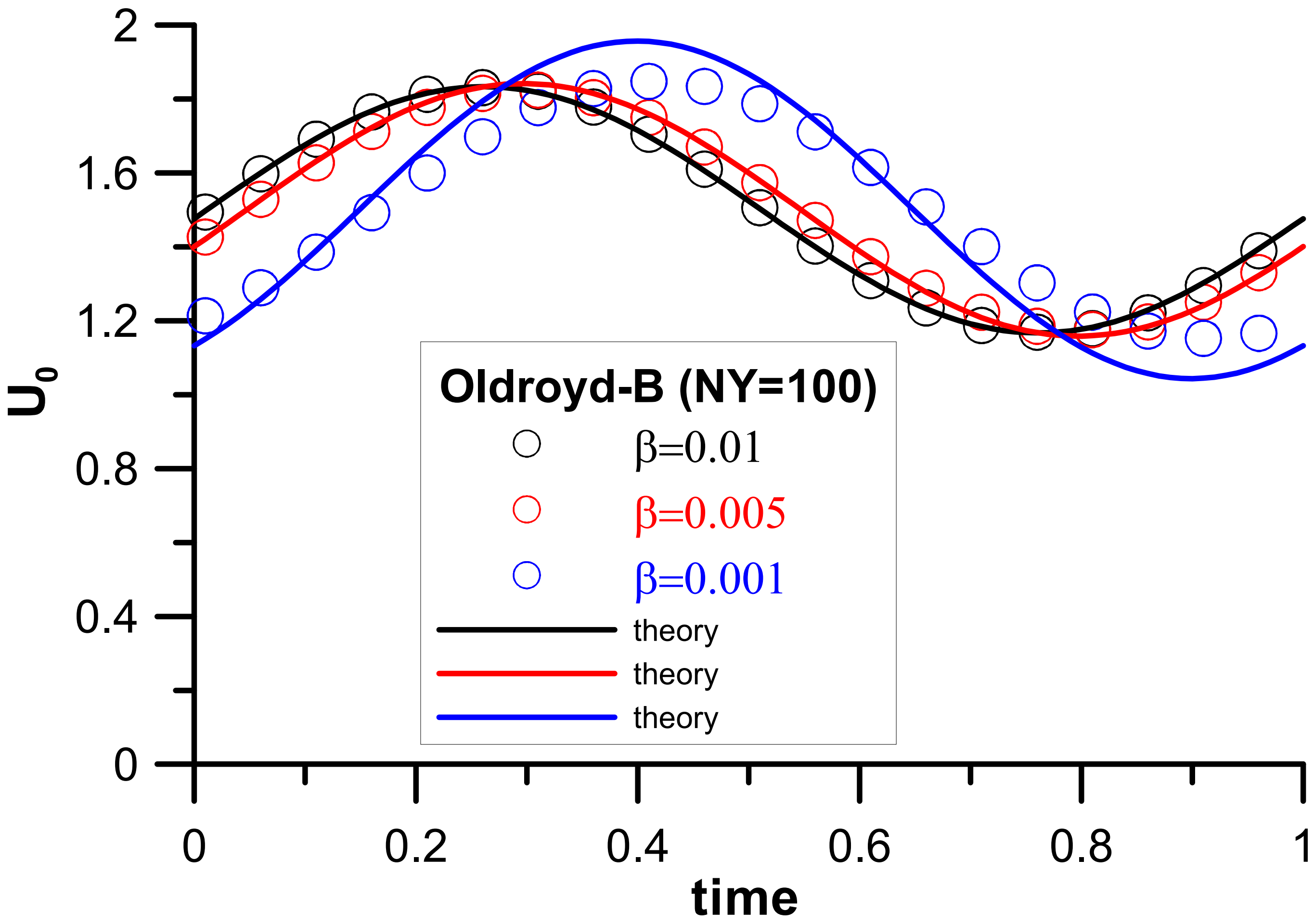
Shear stress profiles

$$T_{xy} = \frac{\tau_{xy}}{\eta_0 \overline{u}_e / h}$$

RESULTS FOR OLDROYD-B

$$\boxed{\beta \in [0.1 - 0.001]}$$

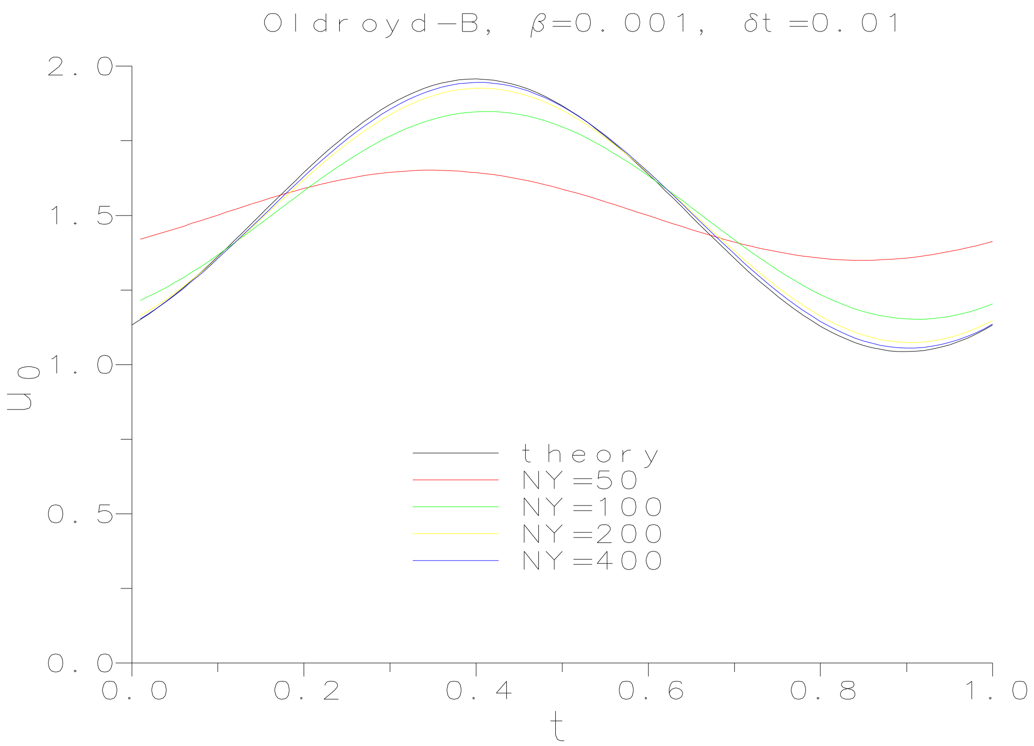
OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT



Centerline velocity: theory and predictions

OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

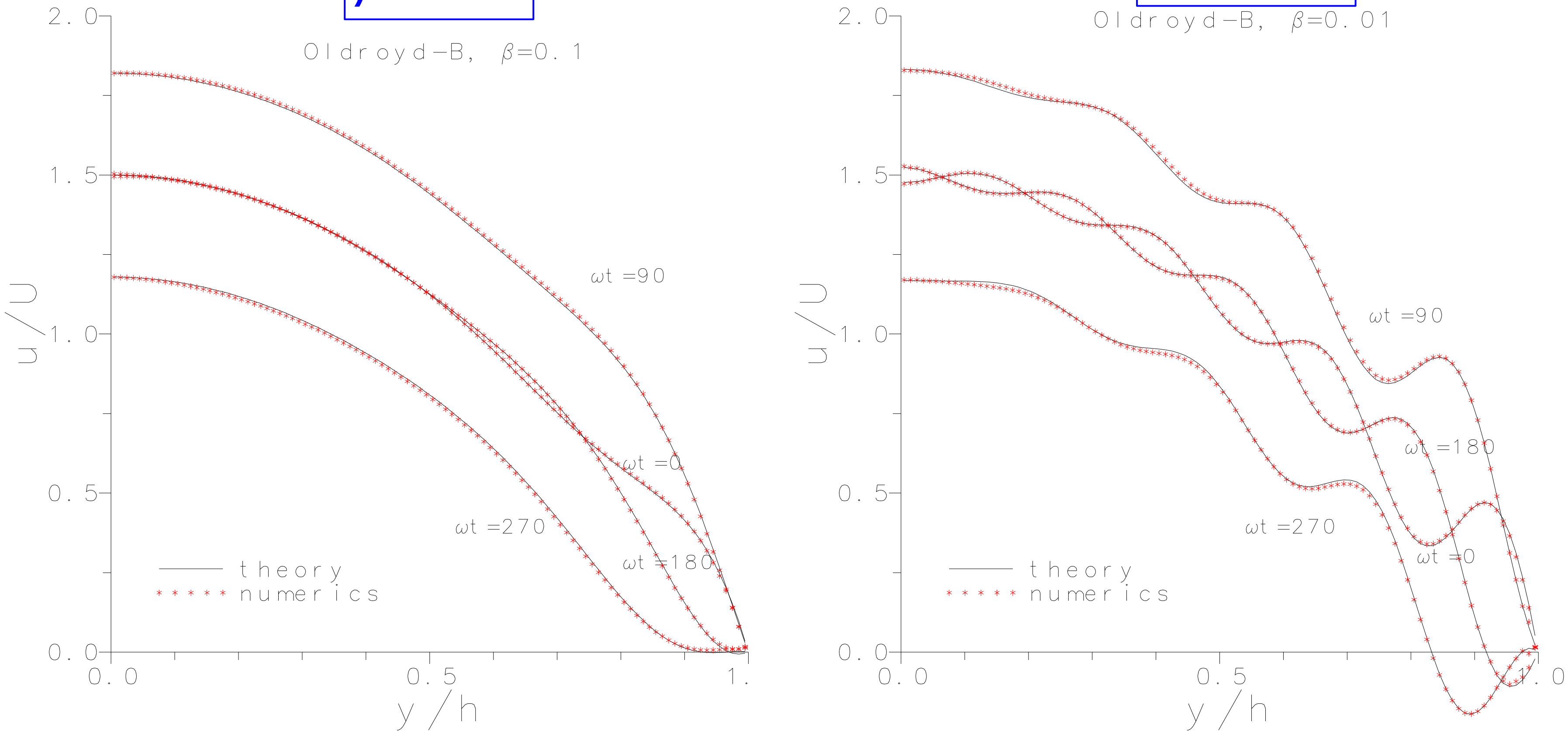
$$\boxed{\beta = 0.001}$$



Centerline velocity: theory and predictions

OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

$\beta = 0.1$ $\beta = 0.01$

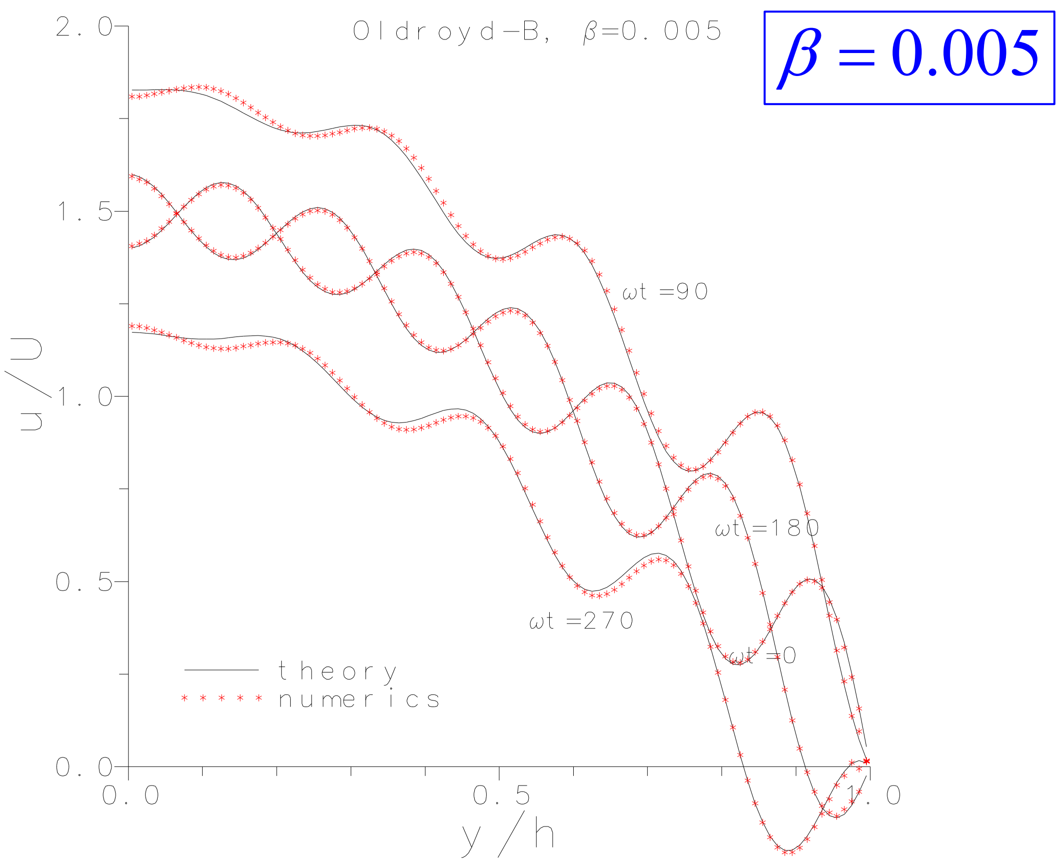


Velocity profiles

$$\beta = \lambda_2 / \lambda_1 = \eta_2 / \eta_0$$

Mesh:NY=100; time step:dt=0.01

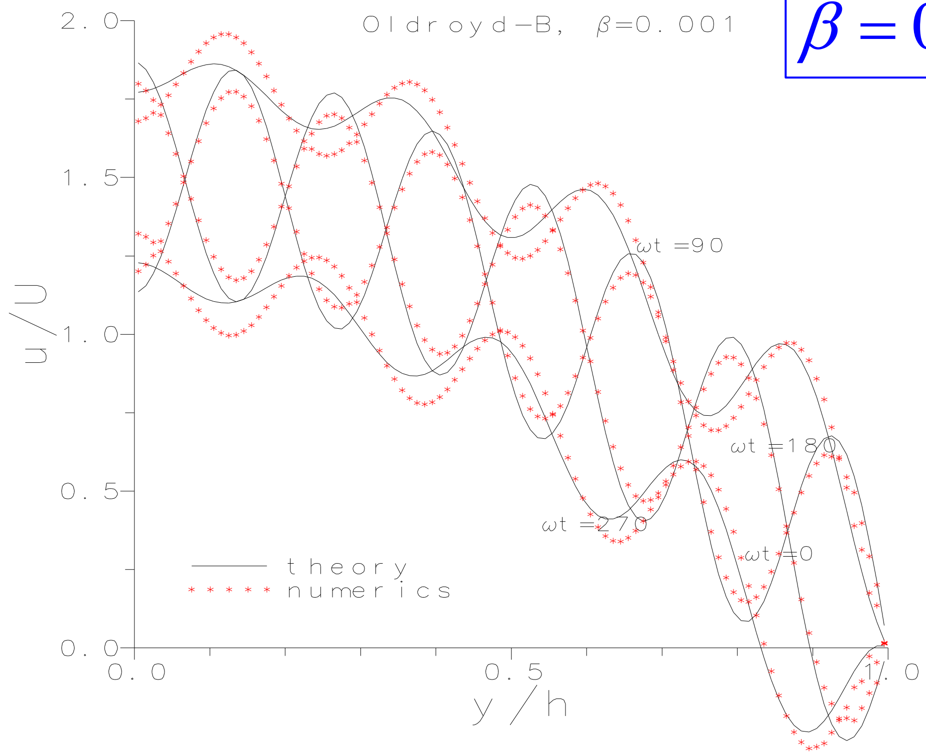
OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT



Velocity profiles

OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

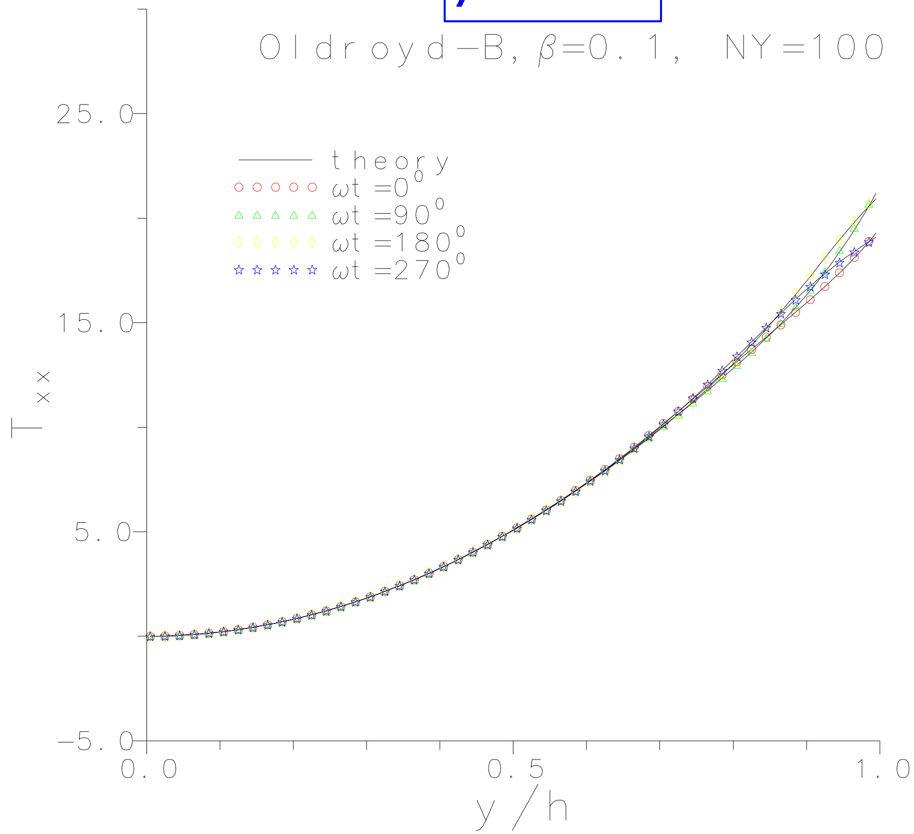
$$\beta = 0.001$$



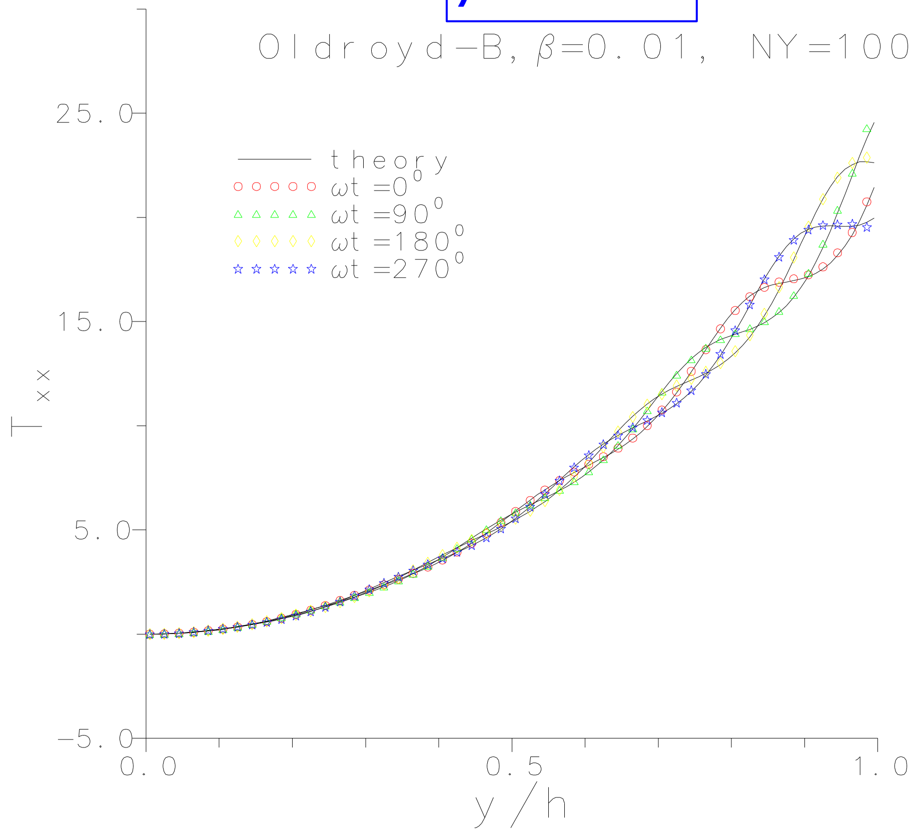
Velocity profiles

OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

$\beta = 0.1$



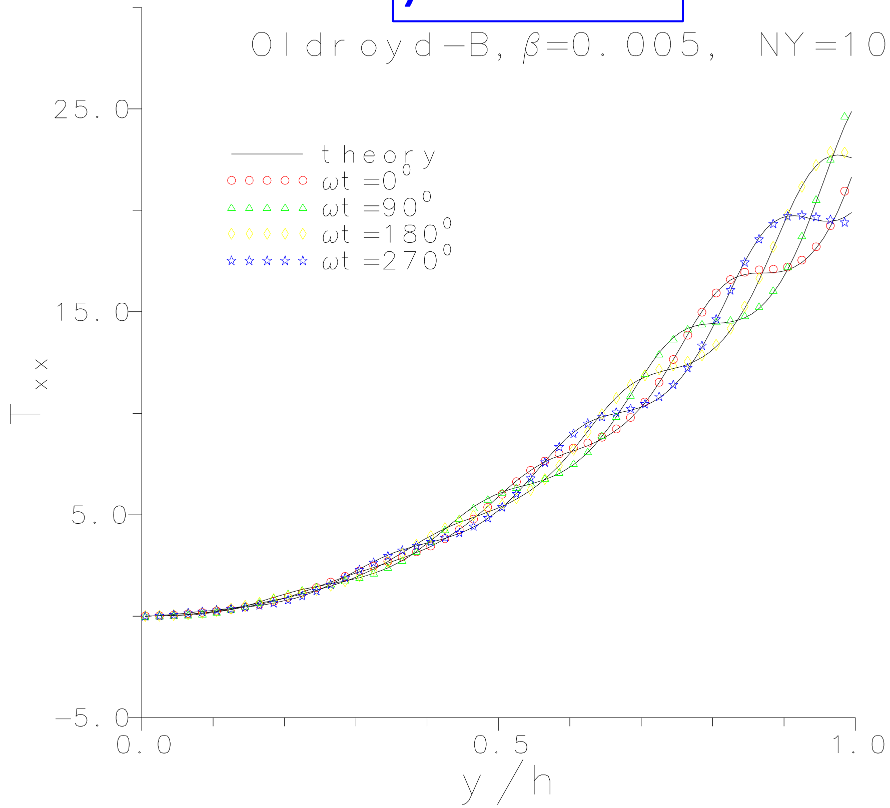
$\beta = 0.01$



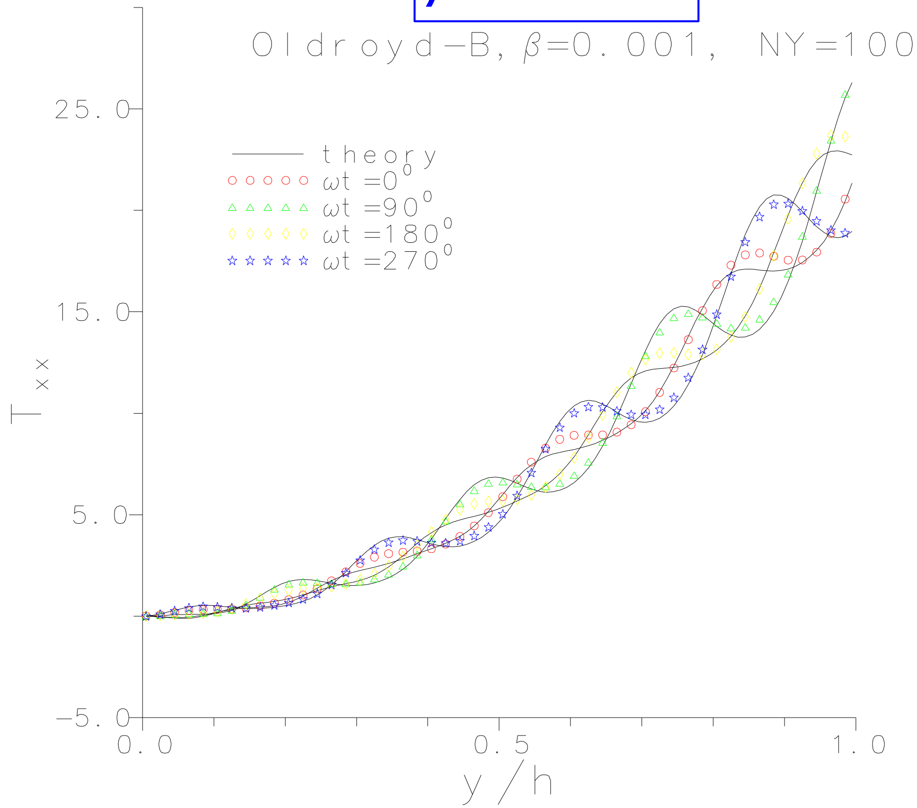
Normal stress profiles

OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

$\boxed{\beta = 0.005}$



$\boxed{\beta = 0.001}$



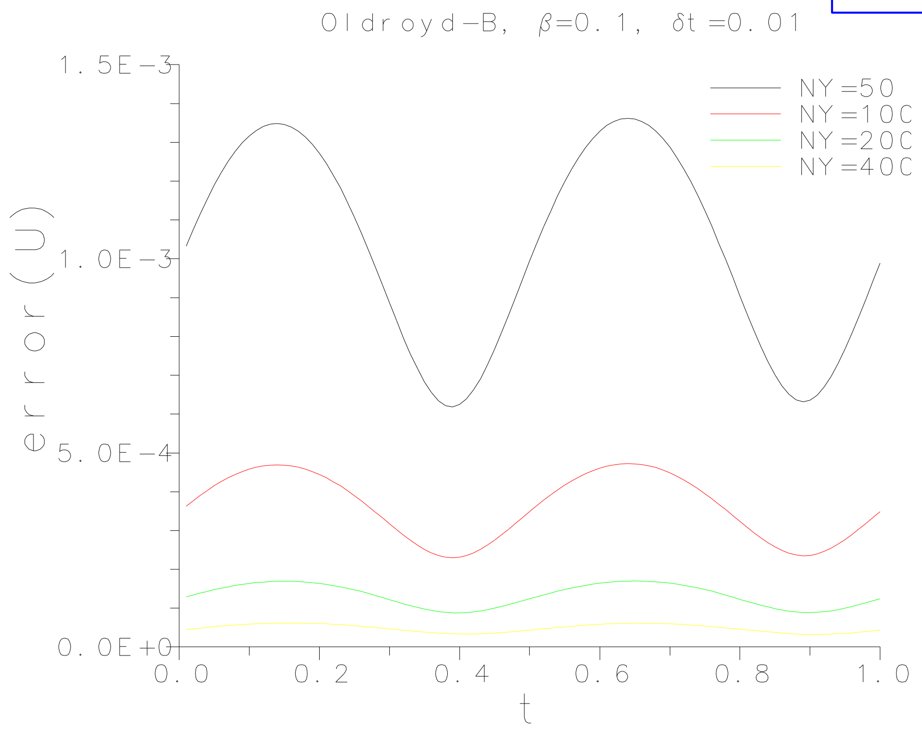
Normal stress profiles

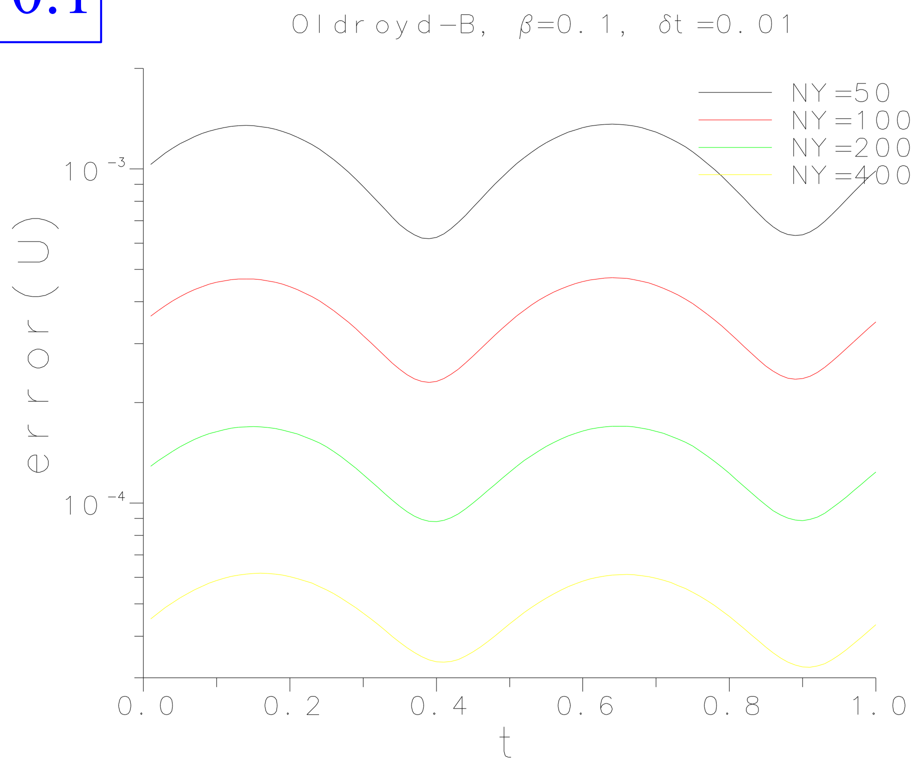
OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

linear scale

$\boxed{\beta = 0.1}$

log scale



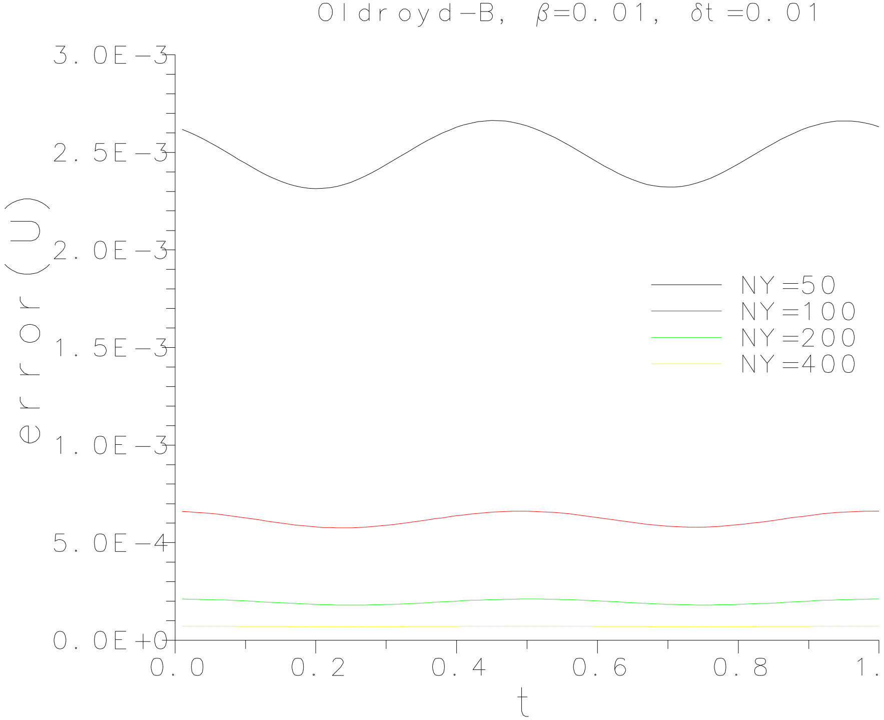


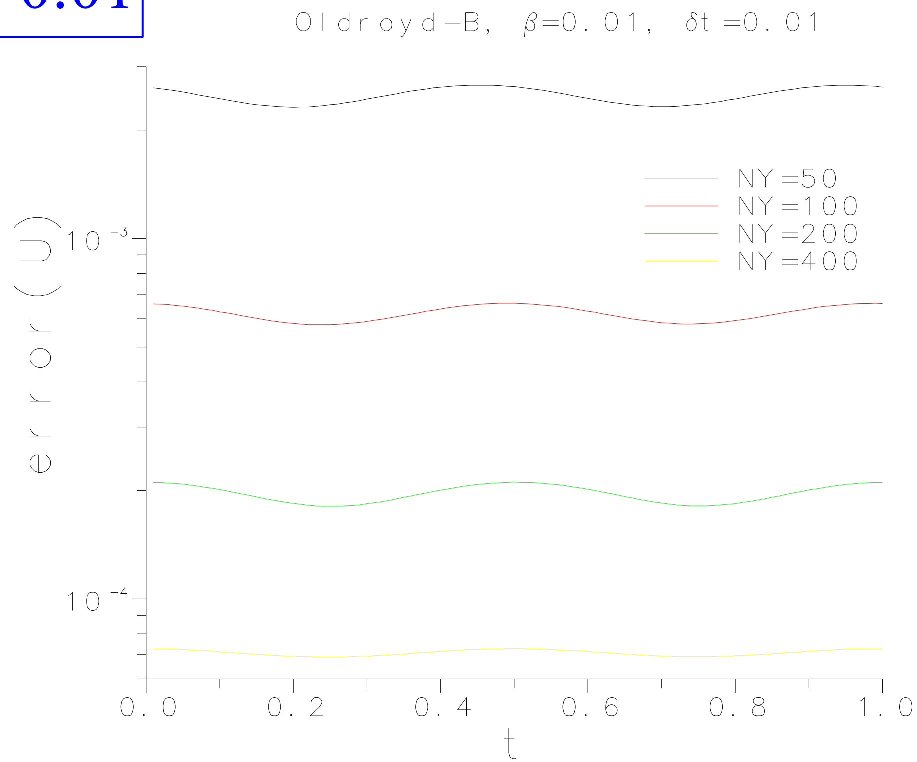
Discretization error for velocity

$$\boxed{e(t) = \frac{1}{NY}\sqrt{\sum_{i=1}^{NY}\left(u_i - u(y_i,t)\right)^2}}$$

OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

$$\boxed{\beta = 0.01}$$

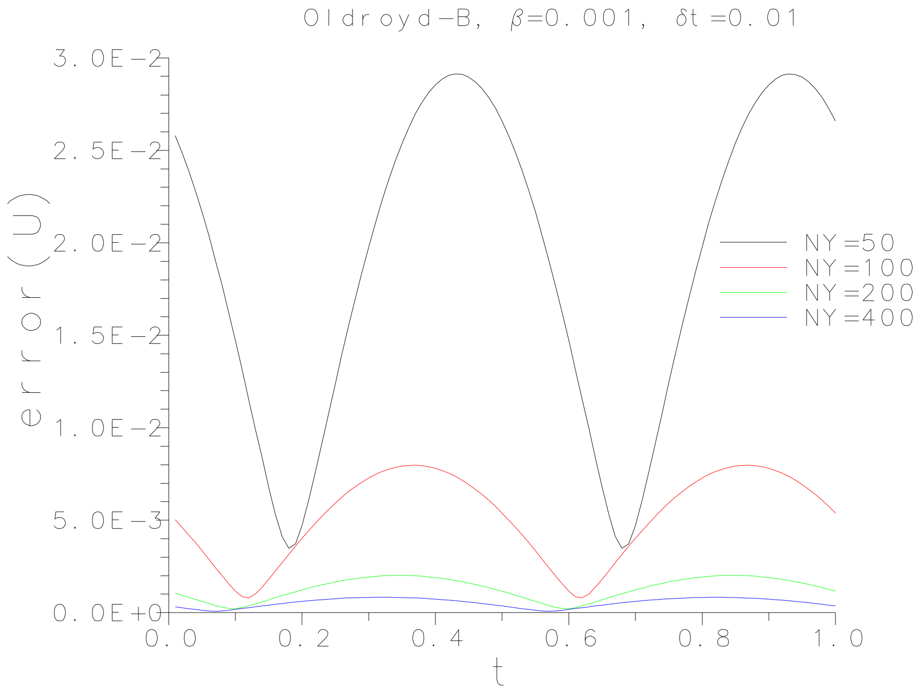


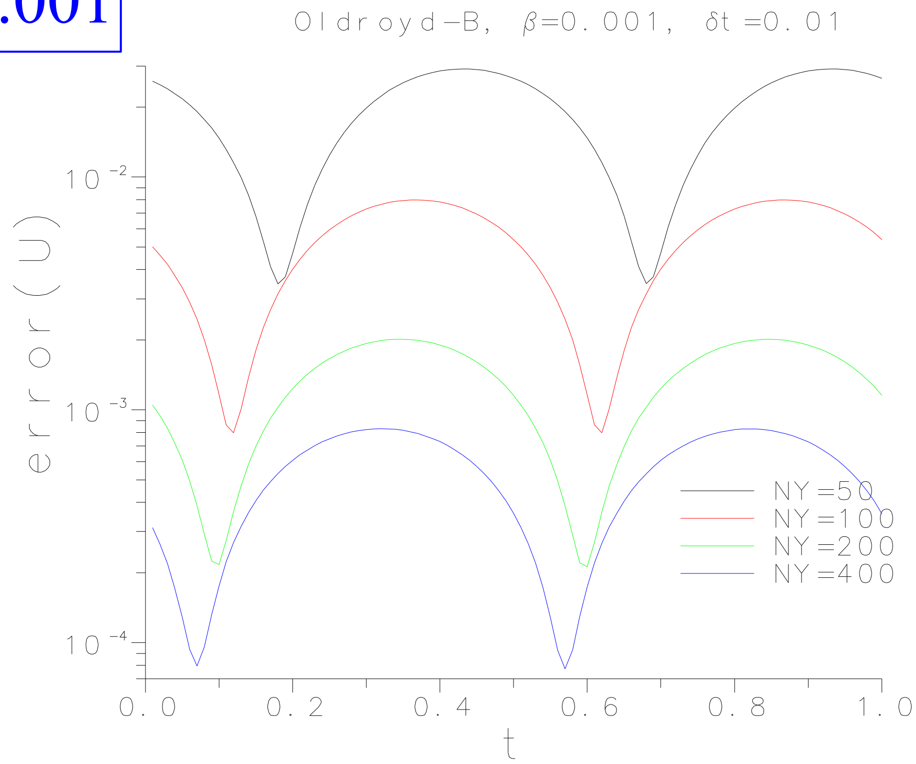


Discretization error for velocity

OLDROYD-B FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

$$\boxed{\beta = 0.001}$$



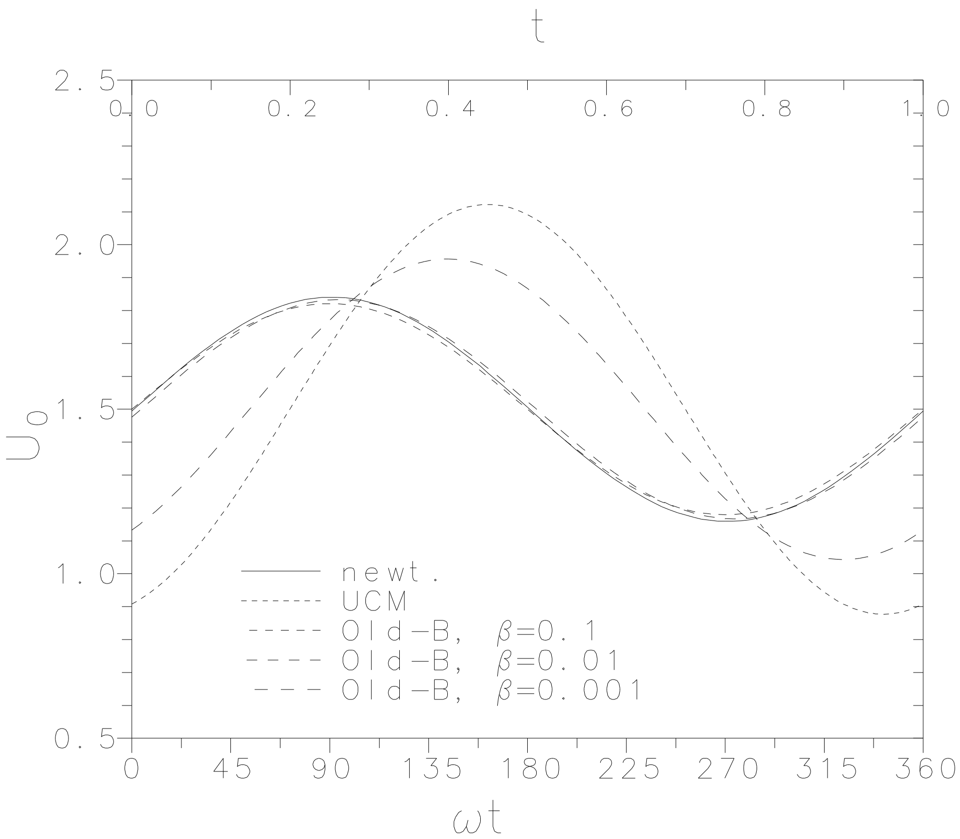


Discretization error for velocity

RESULTS FOR UCM

$$\boxed{\beta \rightarrow 0}$$

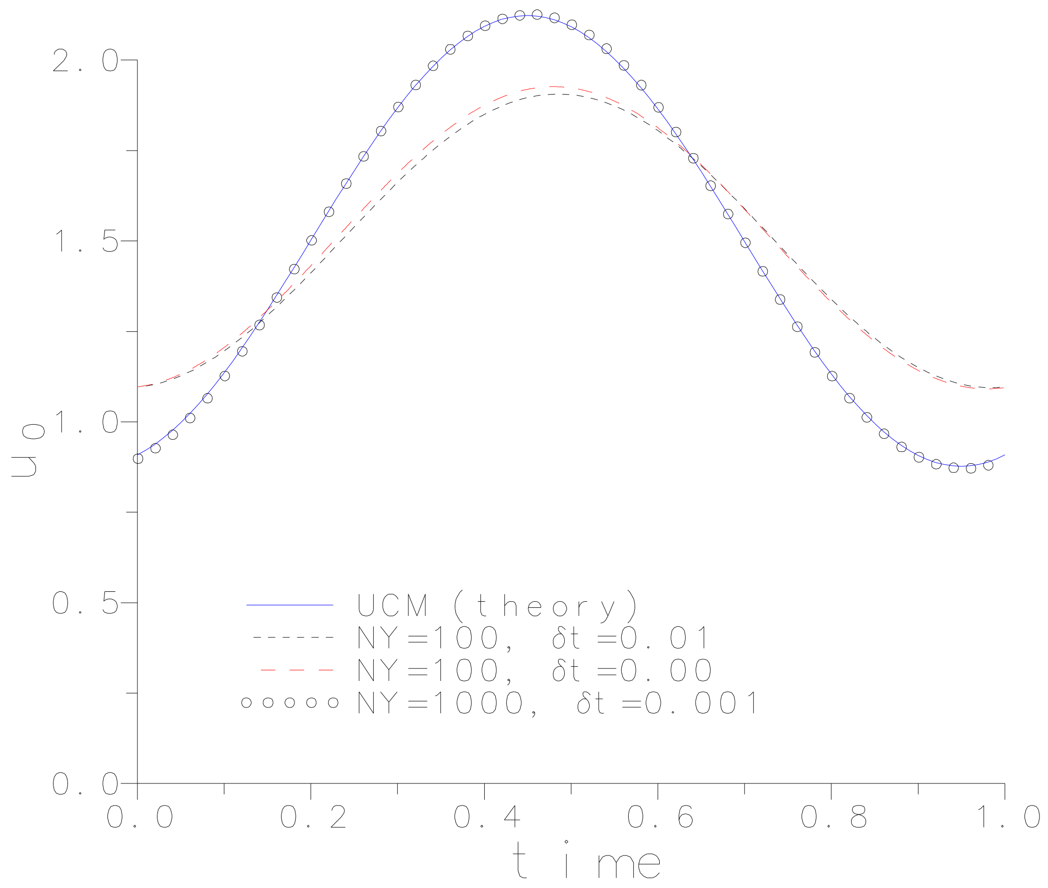
UCM, OLDROY-B AND NEWTONIAN – CENTERLINE VELOCITY



Analytical solution

Note: only for small beta is UCM approached

UCM FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT



Centerline velocity: mesh and dt refinement

UCM FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT

Elasticity number, E=1
Womersley number, α=4.9

$$\alpha = h\sqrt{\frac{\omega}{\eta_o / \rho}}$$

VELOCITY PROFILES

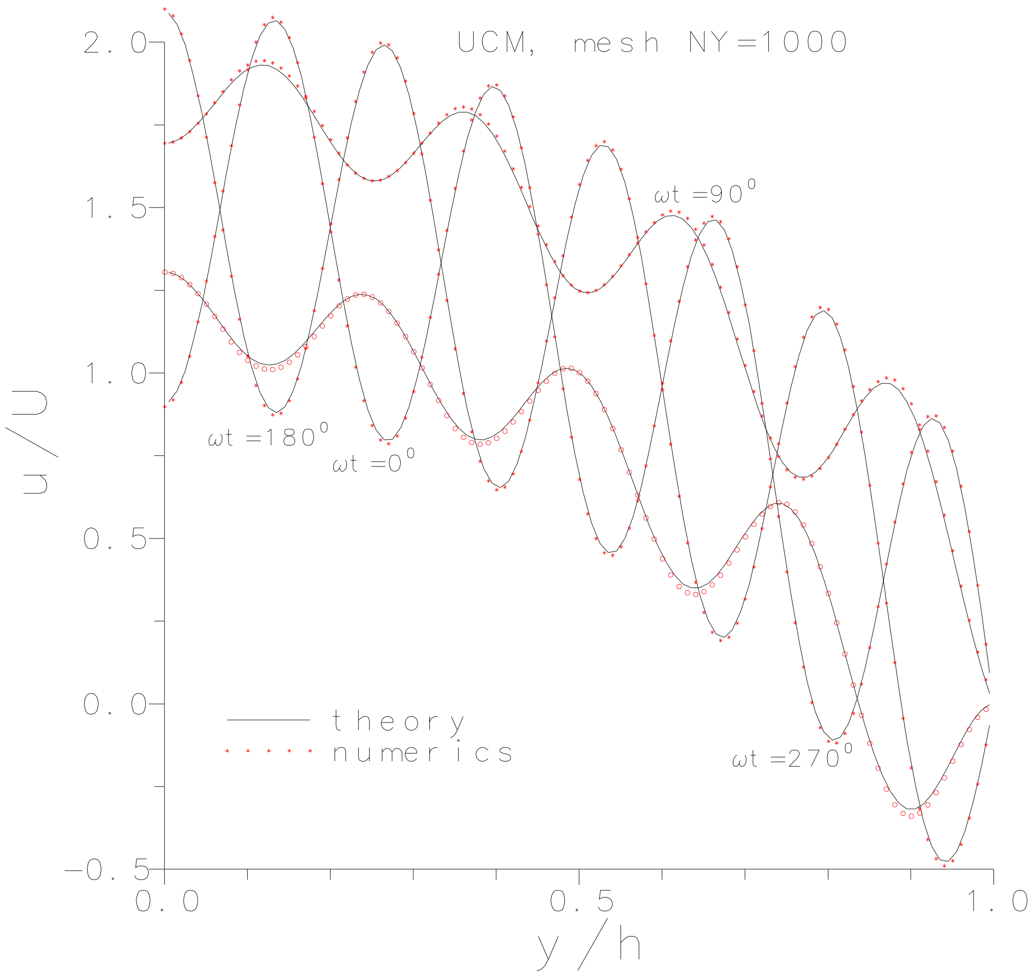
Comparison with theory

Simulation results for a period

Line: simulations

Symbols: theory

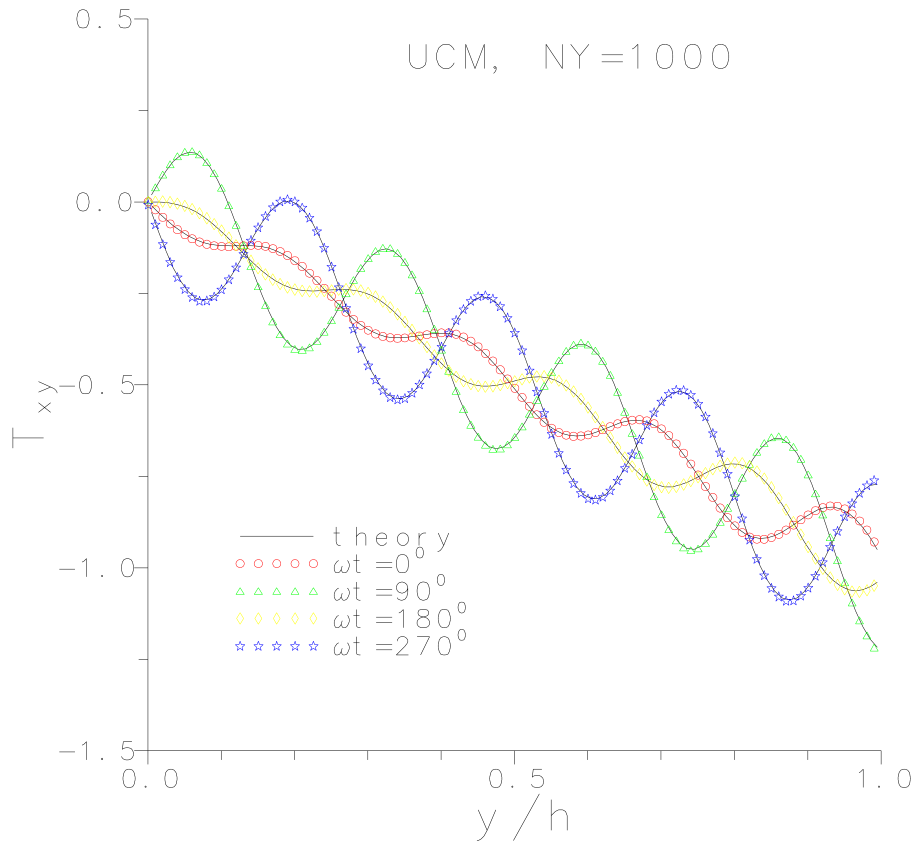
UCM FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT



$$U = \frac{u}{\overline{u}_e}$$

Velocity profiles: NY=1000, dt=0.001

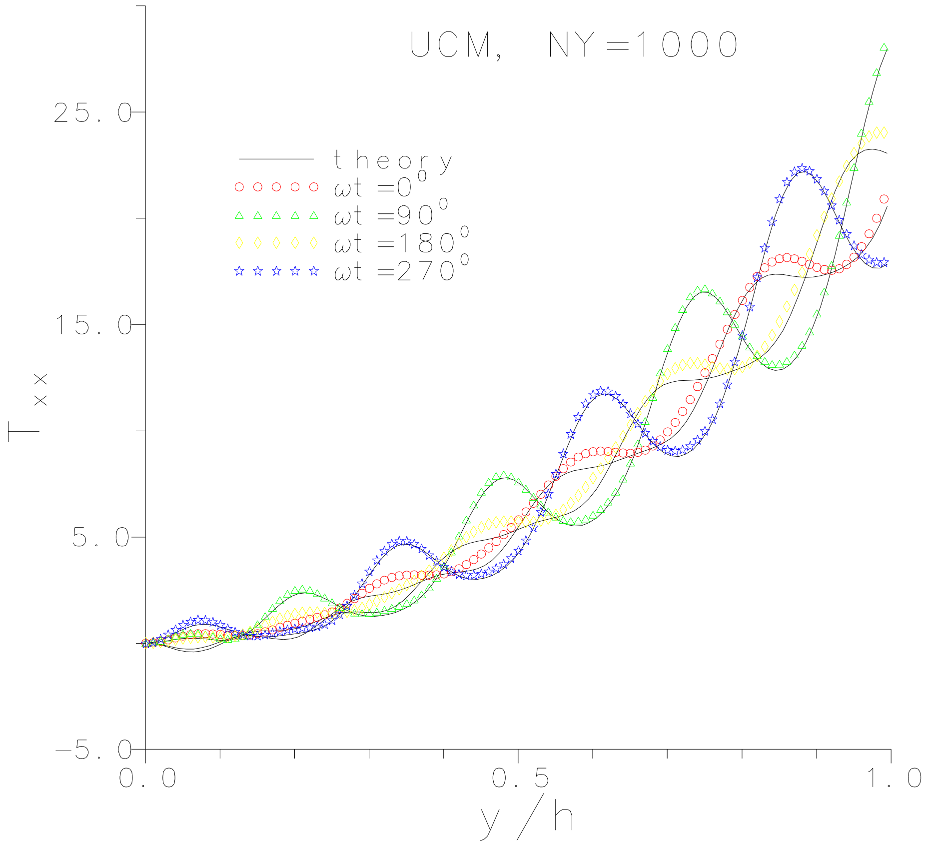
UCM FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT



$$T_{xy} = \frac{\tau_{xy}}{3\eta_0 \bar{u}_e / h}$$

Shear stress profiles

UCM FLOW IN CHANNEL DUE TO PULSATING PRESSURE-GRADIENT



$$T_{xy} = \frac{\tau_{xx}}{3\eta_0 \overline{u}_e / h}$$

Normal stress profiles

CONCLUSIONS

- Analytical solution derived for Oldroyd-B
- Full code able to make fully-developed simulations effectively
- For Oldroyd-B, dt=0.01 and NY=100 are good enough
- For UCM, dt=0.001 and NY=1000 are required
- Discretization error can be computed exactly

Questions & further work

- Are factorisation schemes, without iteration inside dt, adequate?
- Study benefits or not of using EVSS, instead of stress decomposition

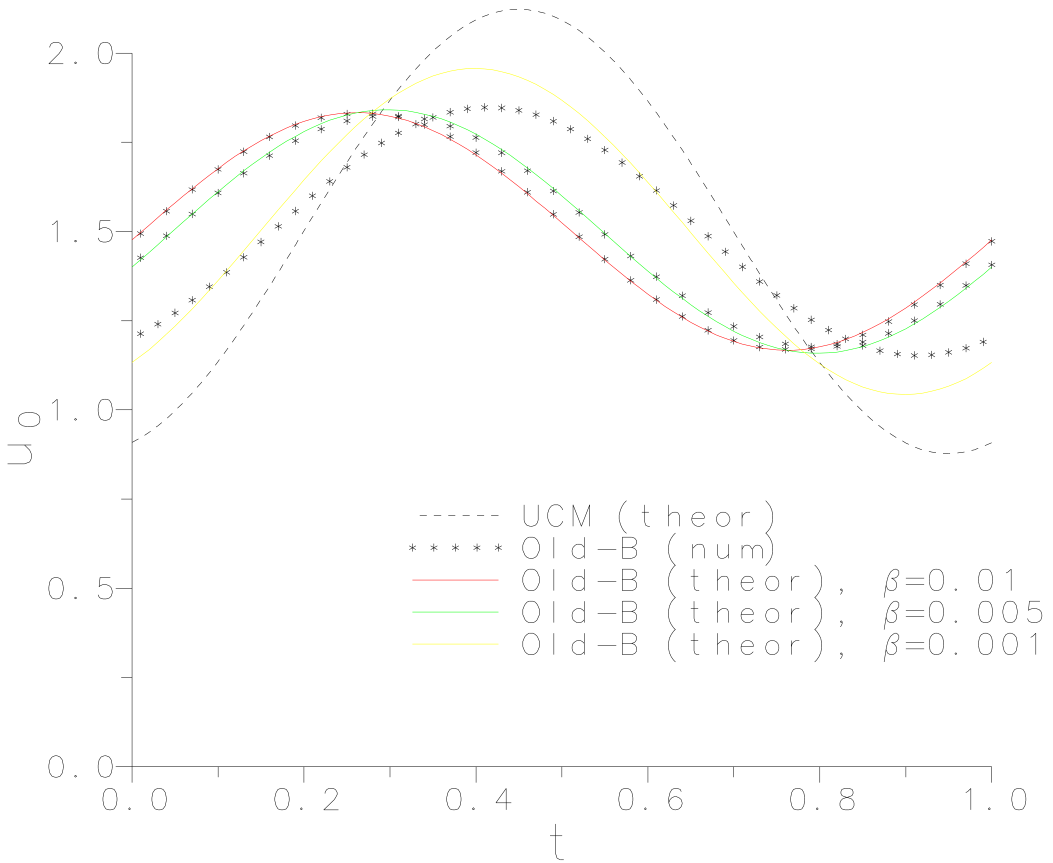
ACKNOWLEDGMENTS:

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, INOVAÇÃO E DO ENSINO SUPERIOR Portugal

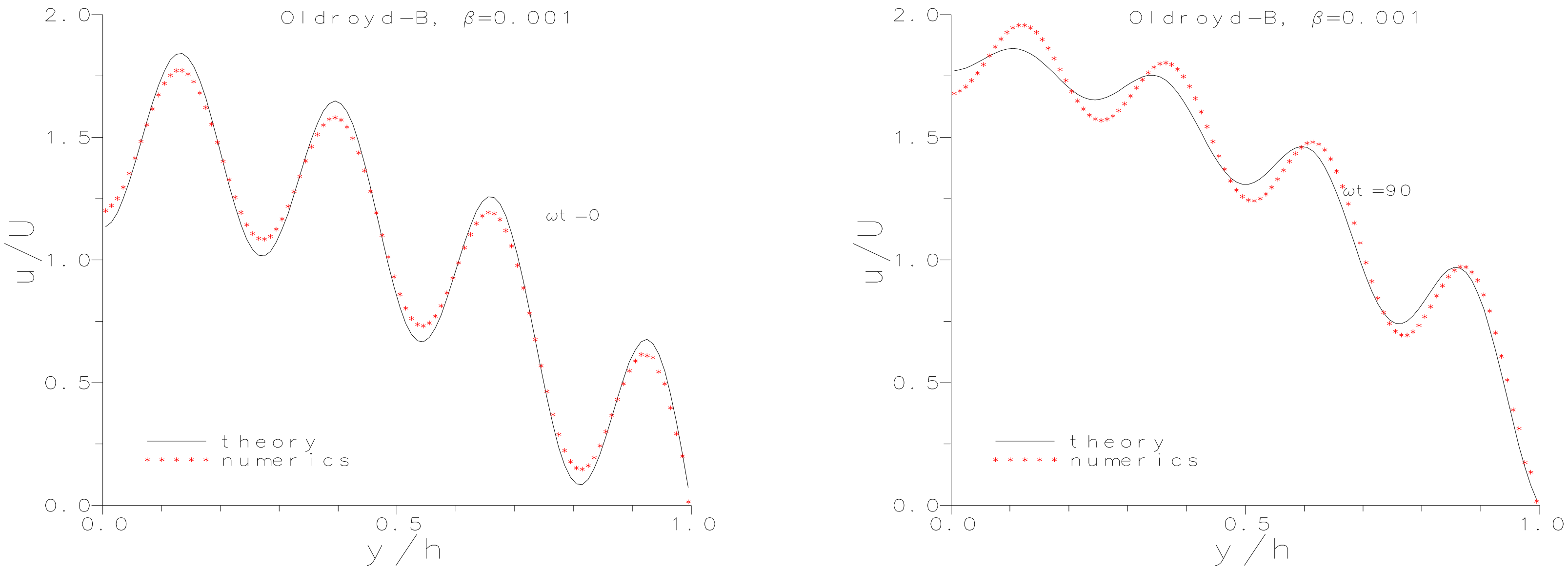
Project POCTI/EME/48665/2002

THANK YOU...

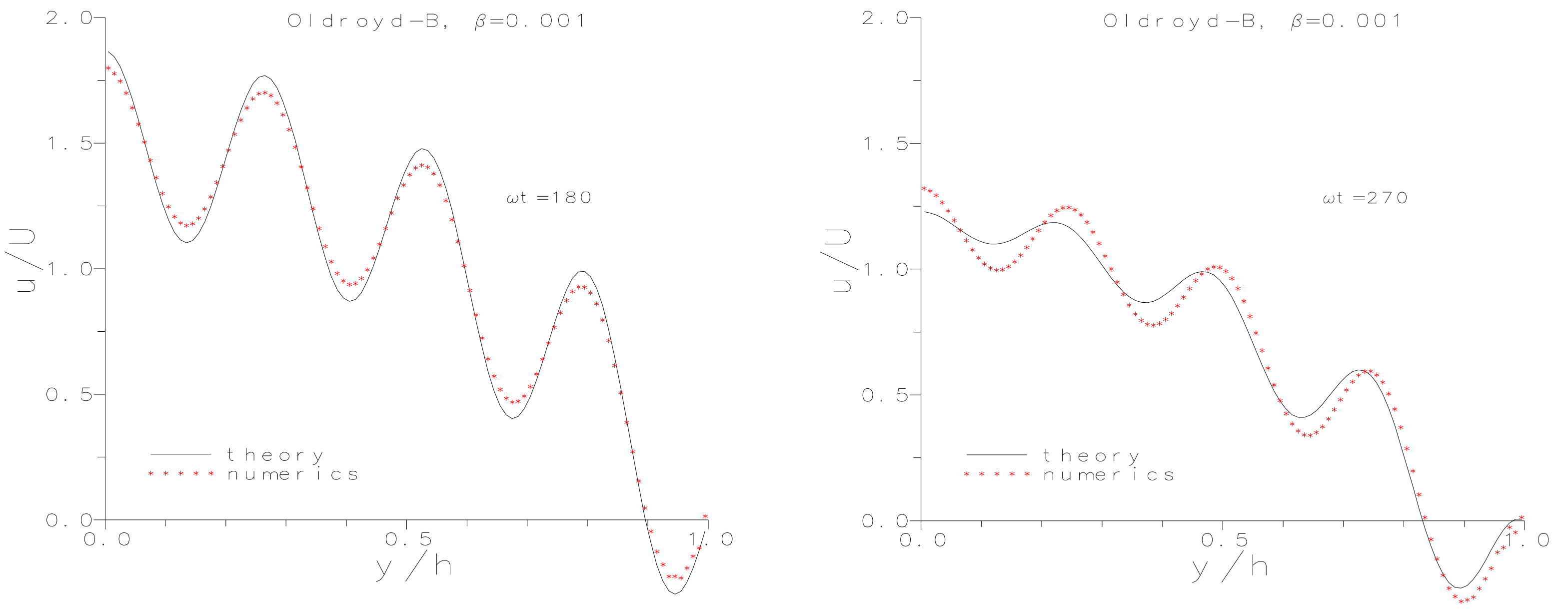
OLDROYD-B FLOW: CENTRELINE VELOCITY, NUMERICAL AND THEORY



OLDROYD-B FLOW: VELOCITY PROFILES FOR BETA=0.001, NY=100, dt=0.01



OLDROYD-B FLOW: VELOCITY PROFILES FOR BETA=0.001, NY=100, dt=0.01



ANALYTICAL SOLUTION FOR PULSATING CHANNEL FLOW OF OLDROYD-B FLUID

$$\alpha^2 = \rho h^2 \omega / \eta_0 \qquad E = \lambda_1 \eta_0 / \rho h^2 \qquad \beta = \lambda_2 / \lambda_1 = \eta_s / \eta_0$$

$$-\frac{dp}{dx} = \rho K_e + \rho K_o \cos(\omega t)$$

$$u(y,t) = \frac{h^2 \rho K_e}{2\eta_0}\left[1-(y/h)^2\right] + \frac{K_o}{\omega}\operatorname{Re}\left\{ i e^{i\omega t}\left[\frac{\cosh(zy/h)}{\cosh z} - 1\right]\right\}$$

$$\tau_{xy}(y,t) = -\rho K_e h(1-\beta)\left(\frac{y}{h}\right) - \rho h K_o \operatorname{Re}\left\{ e^{i\omega t}\frac{\sinh(zy/h)}{\cosh z}\left[\frac{1}{z} + \frac{i\beta z}{\alpha^2}\right]\right\}$$

$$\tau_{xx}(y,t) = \frac{2\lambda}{\eta_0}(\rho K_e h)^2(1-\beta)\left(\frac{y}{h}\right)^2 - 2\left(\frac{K_o}{K_e}\right)\frac{\lambda(\rho h K_e)^2}{\rho h^2 \omega}\operatorname{Re}\left\{\frac{e^{i\omega t}}{1+i\omega t}\frac{\sinh(zy/h)}{\cosh z}Y\left[iz(1-2\beta) - \frac{\alpha^2}{z}\right]\right\}$$

$$-2\left(\frac{K_o}{K_e}\right)^2\frac{\lambda(\rho h K_e)^2}{\rho h^2 \omega}\operatorname{Re}\left\{\frac{e^{i2\omega t}}{1+i2\omega t}\left(\frac{\sinh(zy/h)}{\cosh z}\right)^2\left[i - \frac{\beta z^2}{\alpha^2}\right]\right\}$$